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B-Spline Solution and the Chaotic Dynamics of Troesch's Problem

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A B-spline method is presented for solving the Troesch problem. The numerical approximations to the solution are calculated and then their behavior is studied and commenced. The chaotic dynamics exhibited by the solutions of Troesch's problem as they were derived by the decomposition method approximation are examined and an approximate critical value for the parameter λ is introduced also in this study. For the parameter value slightly less than $\lambda \approx 2.2$, the solutions begin to show successive bifurcations, finally entering chaotic regimes at higher λ values. The effectiveness and accuracy of the B-spline method is verified for different values of the parameter, below its critical value, where the first bifurcation occurs.

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1. Introduction

We consider the Troesch problem

$$u''(x) = \lambda \sinh(\lambda u(x)), \quad 0 < x < 1 \tag{1}$$

with the boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(1) = 1. \tag{2}$$

It is an unstable two point boundary value problem (BVP) [1, 2] and arises from a system of nonlinear ordinary differential equations which occur in an investigation of the confinement of a plasma column by radiation pressure [3]. Robert and Shipman showed that this equation can be solved for $\lambda < 5$ [1].

Many different techniques, such as decomposition method [3], homotopy perturbation technique [4], Laplace transform decomposition method [5], differential transform method [6], variational iteration method [7], initial value method [1], Adomian decomposition method [8], validating solver for parametric ordinary differential equations (ODEs) (VSPODE) [9] have been used to solve the Troesch problem. They have also shown that the numerical results do not converge to sufficient accuracy for $\lambda > 1$ [3, 4]. References [3–8] can solve Troesch's problem for the small values of λ.

Troesch's problem can be solved with higher parameters of λ [10–13]. For instance, the finite difference method is used for solving two point BVPs including Troesch's problem for $\lambda = 10$ [10–12]. They have shown that the true solution to this problem has a very sharp boundary layer near $x = 1$ [10–13]. Only in Ref. [13], a finite difference method is used. The others used an invariant imbedding algorithm combined with quasilinearization, an orthonormalizations algorithm combined with quasilinearization, and a multiple shooting technique, respectively.

In the present paper we intend to use the B-spline method for solving the problem. This method has been implemented with success in a nonlinear BVP [14]. In this work, third degree B-spline functions have also been successfully implemented to Troesch's problem. The obtained results are numerically compared with corresponding results from other methods given in literature [4].

In order to understand the chaotic dynamics of the Troesch problem, the decomposition method approximate solutions

$$u_1(x) = \frac{\sinh(\lambda x) - x\sinh(\lambda)}{\lambda} \tag{3}$$

and

$$\begin{aligned} u_2(x) = -\frac{1}{4\lambda^2}\big[&-\lambda\cosh(\lambda x)\sinh(\lambda x) \\ &+ 4\lambda x\sinh(\lambda)\cos(\lambda x) - 8\sinh(\lambda)\sinh(\lambda x) \\ &- 3\lambda x\sinh(\lambda)\cos h(\lambda) + 8x\cosh^2(\lambda) - 8x\big] \end{aligned} \tag{4}$$

are used [3].

We examined the chaotic dynamics exhibited by the solutions. It was found that for certain values of the parameter λ, it undergoes successive bifurcations and shows distinctive chaotic regimes. The observed bifurcation behavior has been found and studied by Roberts and Shipman [15]. They have shown diagrams of continuous and discontinuous solutions. They have also compared their results with the results of Ref. [16] where the continuous and discontinuous solutions exist. In particular for different values of λ we can verify the following:

for $< \lambda_c$, the problem has a unique solution,

for $> \lambda_c$, the problem has several solutions in chaotic region, where the critical value is $\lambda_c \approx 2.2$.

2. The third-degree B-splines

In this section, third-degree B-splines are used to construct numerical solutions to Troesch's problem that is

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given in Eqs. (1) and (2). A detailed description of B-spline functions can be found in [17]. The third-degree B-splines are defined as:

$$B_0(x) = \frac{1}{6h^3}\begin{cases} x^3, \quad 0 \le x < h, \\ -3x^3 + 12hx^2 - 12h^2x + 4h^3, \\ \quad h \le x < 2h, \\ 3x^2 - 24hx^2 + 60h^2x - 44h^3, \\ \quad 2h \le x < 3h, \\ -x^3 + 12hx^2 - 48h^2x + 64h^3, \\ \quad 3h \le x < 4h, \end{cases}$$

$$B_{i-1}(x) = B_0\left[x - (i-1)h\right], \quad i = -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots \tag{5}$$

To solve nonlinear Troesch's problem, B_i, B_i', B_i'' evaluated at the nodal points are needed. Their coefficients are summarized in Table I.

TABLE I

Values of $B_i B_i'$ and B_i''.

	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
B_i	0	1	4	1	0
B_i'	0	$-3/h$	$0/h$	$3/h$	0
B_i''	0	$6/h^2$	$-12/h^2$	$6/h^2$	0

3. B-spline solutions for Troesch's problem

Let

$$S(x) = \sum_{j=-3}^{n-1} C_j B_j(x) \tag{6}$$

be an approximate solution of Eqs. (1) and (2), where C_j are unknown real coefficients and $B_j(x)$ are third-degree B-spline functions. Let $x_0, x_1, \ldots, x_n$ be $n+1$ grid points in interval $[a, b]$ so that $x_i = a + ih$, $i = 1, 2, \ldots n$, $x_0 = a$, $x_n = b$, $h = (b-a)/n$. The approximate solution (6) is substituted in Eqs. (1) and (2) and evaluated at the grid points $x_0, x_1, \ldots, x_n$. This leads to a nonlinear system of equations of the form

$$\sum_{j=-3}^{n-1} C_j B_j''(x_i) - \lambda \sinh\left(\lambda \sum_{j=-3}^{n-1} C_j B_j(x_i)\right) = 0, \quad i = 0, 1, \ldots, n, \tag{7}$$

$$\sum_{j=-3}^{n-1} C_j B_j(x_i) = 0, \quad \text{for } x = 0, \tag{8}$$

$$\sum_{j=-3}^{n-1} C_j B_j(x_i) = 1, \quad \text{for } x = 1. \tag{9}$$

The values of the spline functions at the knots $\{x_i\}_{i=0}^{n}$ are determined using Table I with substitution in Eqs. (7)–(9). Thus a system of $n+3$ nonlinear equations in the $n+3$ unknowns $C_3, C_2 \ldots, C_{n-1}$ is obtained. This system may be written in matrix-vector form as follows:

$$AC - \lambda B = 0, \tag{10}$$

where

$$A = \begin{bmatrix} 1 & 4 & 1 & 0 & 0 & 0 & \ldots & 0 \\ \frac{6}{h^2} & \frac{-12}{h^2} & \frac{6}{h^2} & 0 & 0 & 0 & \ldots & 0 \\ 0 & \frac{6}{h^2} & \frac{-12}{h^2} & \frac{6}{h^2} & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \frac{6}{h^2} & \frac{-12}{h^2} & \frac{6}{h^2} \\ 0 & 0 & 0 & 0 & \ldots & 1 & 4 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ \sinh(\lambda\,(C_{-3} + 4C_{-2} + C_{-1})) \\ \sinh(\lambda\,(C_{-2} + 4C_{-1} + C_0)) \\ \sinh(\lambda\,(C_{-1} + 4C_0 + C_1)) \\ \vdots \\ \sinh(\lambda\,(C_{n-4} + 4C_{n-3} + C_{n-2})) \\ \sinh(\lambda\,(C_{n-3} + 4C_{n-2} + C_{n-1})) \\ 0 \end{bmatrix}$$

and

$$C = \left[C_{-3}, C_{-2}, C_{-1}, \ldots, C_{n-3}, C_{n-2}, C_{n-1}\right]^{\mathrm{T}}.$$

The approximate solution (Eq. (6)) is obtained by solving the nonlinear system using the Levenberg–Marquardt optimization method [18] and MATLAB 6.5.

4. Chaotic behaviour of Troesch's equation approximate solutions

Troesch's and Bratu's problems belong to a class of nonlinear BVPs. In a previous work we have studied the chaotic dynamics of Bratu's equation in detail [19]. As given in literature, we have also found difficulties in B-spline method for higher values of λ. In order to understand the dynamics of Troesch's problem, the decomposition method approximate solutions are used for the iterations.

A reccurent form of the approximate solutions of Troesch's problem given in Eqs. (3) and (4) are written as

$$u_{n+1,i}(x) = \frac{\sinh\left(\lambda_i u_{n,i}\right) - u_{n,i}\sinh(\lambda_i)}{\lambda_i} \tag{11}$$

and

$$\begin{aligned} u_{n+1,i}(x) = -\frac{1}{4\lambda_i^2}\Big[&- \lambda_i \cosh\left(\lambda_i u_{n,i}\right)\sinh\left(\lambda_i u_{n,i}\right) \\ &+ 4\lambda_i u_{n,i}\cosh\left(\lambda_i u_{n,i}\right)\sinh\left(\lambda_i\right) \\ &- 8\sinh\left(\lambda_i\right)\sinh\left(\lambda_i u_{n,i}\right) - 3\lambda_i\cosh\left(\lambda_i\right)\sinh\left(\lambda_i\right) \\ &+ 8u_{n,i}\cosh^2\left(\lambda_i\right) - 8u_{n,i}\Big]. \end{aligned} \tag{12}$$

In a next step, we are creating the graph of u_n versus x. To avoid initial fluctuations we performed the averaging over the last 100 values of 2,000 iterations. For this purpose we have used MATHCAD to calculate the bifurcation diagrams (Figs. 1 and 2).

To verify the critical values of λ where bifurcations occur and the onset of chaos, a plot of the calculated Lyapunov exponents versus the parameter is given in Figs. 3 and 4 for the bifurcation diagrams given in Figs. 1 and 2,

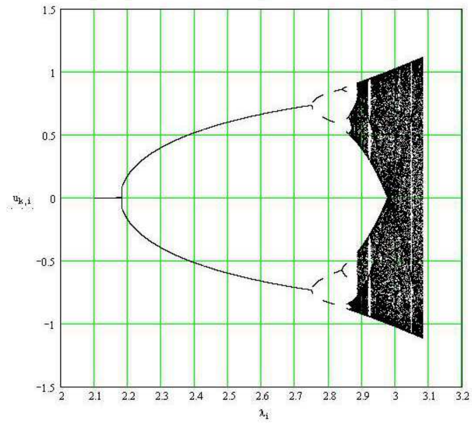

Fig. 1. u_n plotted versus λ for the first approximate solution.

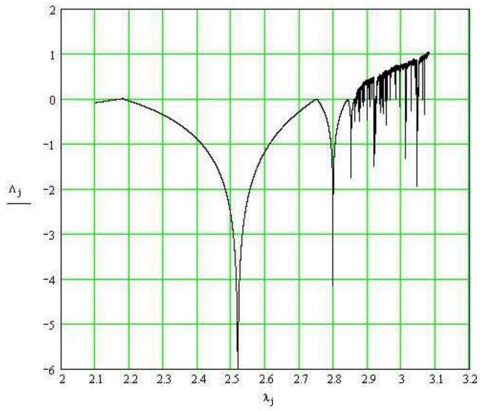

Fig. 2. Lyapunov exponents Λ versus λ for the first approximate solution. Note that Λ becomes zero for the same values of λ, which correspond to bifurcations of the previous graph, while it becomes positive in the chaos regime of the previous graph.

respectively. For this purpose MATHCAD is used again. Doing so, we took into account that the Lyapunov exponent can be estimated using the formula

$$\Lambda \approx \lim_{N\to\infty} \sum_{n=1}^{N} \ln\left(\left|\frac{\mathrm{d}u_{n+1}}{\mathrm{d}u_n}\right|\right). \qquad (13)$$

In the case of our map, it becomes

$$\Lambda \approx \frac{1}{N}\sum_{n=1}^{N} \ln\left(\left|\frac{\lambda\cosh(u_n\lambda)-\sinh(\lambda)}{\lambda}\right|\right) \qquad (14)$$

and

$$\Lambda \approx \frac{1}{N}\left[\sum_{n=1}^{N} \ln\left(\left|\frac{-1}{4\lambda^2}[-\lambda^2\sinh(u_n\lambda)^2 - \lambda^2\cosh(u_n\lambda)^2 + 4\lambda\sinh(\lambda)\cosh(u_n\lambda) + 4u_n\lambda^2\sin(\lambda)\sinh(u_n\lambda) - 8\lambda\sinh(\lambda)\cosh(u_n\lambda) - 3\lambda\sinh(\lambda\cosh(\lambda) + 8\cosh(\lambda)^2 - 8\right|\right)\right]. \qquad (15)$$

For the calculations of the Lyapunov exponents Λ we have used Eqs. (14) and (15). To avoid initial fluctuations we performed the averaging over the last 200 values of 2000 iterations. As clearly seen in the both graphs, the Lyapunov exponents are zero where the parameter $\lambda \approx 2.2$. This signifies that these parameters give an orbit which is on the edge of order and disorder where the bifurcation starts.

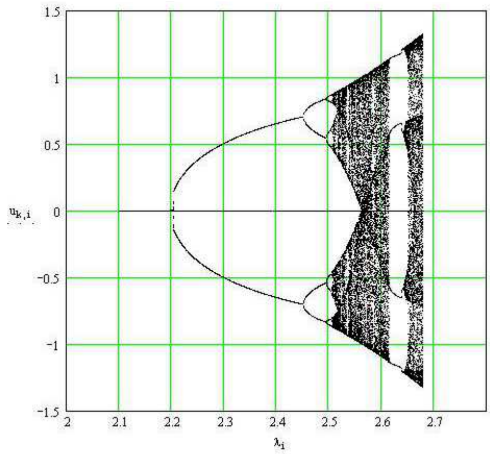

Fig. 3. u_n plotted versus λ for the second approximate solution.

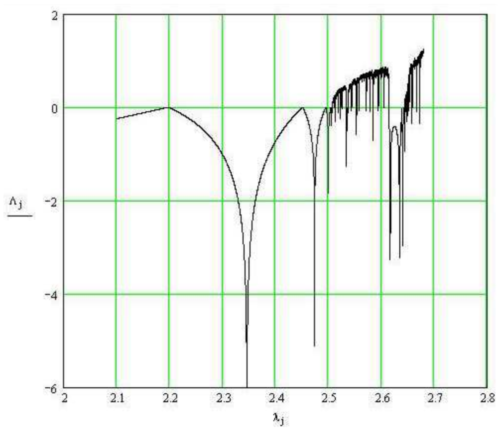

Fig. 4. Lyapunov exponents Λ versus λ for the first approximate solution. Note that Λ becomes zero for the same values of λ, which correspond to bifurcations of the previous graph, while it becomes positive in the chaos regime of the previous graph.

5. Numerical results

In this section, we illustrate the numerical techniques discussed previously, by applying our method to the Troesch problem for two specific values of λ, which guarantee the existence of two locally unique solutions. In particular, having used $\lambda = 0.5$ and 1.0, we have constructed comparison tables (at the end of paper) to indicate the accuracy of our method compared with the exact solution as well as the other methods' solutions. The numerical results are shown in Fig. 5 for $n = 21$, the number of mesh points.

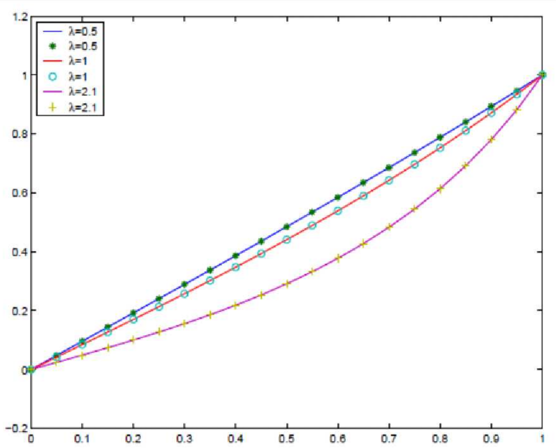

Fig. 5. Results for $\lambda = 0.5, 1$ and 2.1 $(n = 21)$.

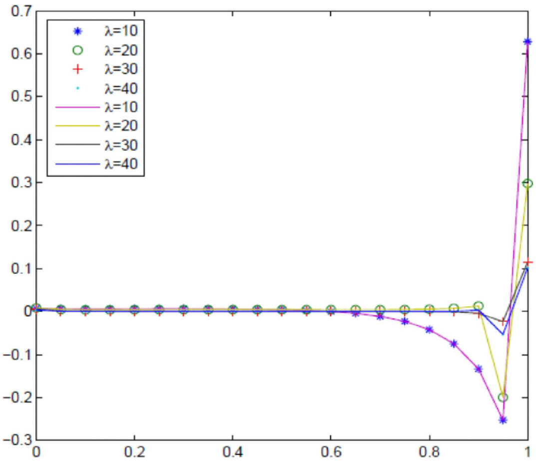

Fig. 6. Results for $\lambda = 10, 20, 30$ and 40 $(n = 21)$.

In Tables II and III, we have tabulated the exact solution together with the solutions given by algorithm, variational iteration, Adomian decomposition, and B-spline methods are as exhibited before, in the case of $\lambda = 0.5$ and $\lambda = 1.0$, respectively. In Tables IV and V, we have calculated the absolute error of each method compared with the exact solution for $\lambda = 0.5$ and $\lambda = 1.0$, respectively. As one clearly observes, the numerical results using the B-spline method converge to the exact solution with sufficient accuracy for small values of the number of mesh points $(n = 21)$. The effect of the number of mesh points, n, is also shown (for the case $\lambda = 0.5$). As clearly seen in Table VI, the numerical results do not change so much as the value of n increases.

6. Conclusions

In this paper, a B-spline method is developed for the approximate solution of Troesch's problem. The numerical results obtained by using the method described in this study give acceptable results for the parameter $\lambda < 2.2$. We can conclude that the numerical results converge to the exact solution with sufficient accuracy. We found difficulties in B-spline method for higher values of λ (Fig. 6). So, we examined the dynamics exhibited by the decomposition method approximate solutions of Troesch's problem. It was found that for certain values of the parameter λ ($\lambda_c \approx 2.2$), it undergoes successive bifurcations and shows distinctive chaotic regimes. This behaviour was verified by calculation of the corresponding Lyapunov exponents which becomes zero when bifurcations occur and positive when chaos sets on.

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TABLE II

Algorithm, variational iteration, Adomian decomposition and B-spline method approximations of Troesch's problem for the case $\lambda = 0.5$.

x_i	u_{exact}	u_2 [4]	u_{var} [4]	u_{decom} [4]	B-spline
0.1	0.0951769020	0.0959947969	0.1000416771	0.0959477541	0.0959439328
0.2	0.1906338691	0.1922293333	0.2003336671	0.1921352537	0.1981279377
0.3	0.2866534030	0.2889439740	0.3011275394	0.2888034214	0.2887932439
0.4	0.3835229288	0.3863803333	0.4026773946	0.3861955524	0.3861834133
0.5	0.4815373854	0.4847819010	0.5052411788	0.4845585473	0.4845455521
0.6	0.5810019749	0.5843946667	0.6090820555	0.5841442013	0.5841315802
0.7	0.6822351326	0.6854677448	0.7144698597	0.6852105701	0.6851995784
0.8	0.7855717867	0.7882540000	0.8216826569	0.7880234321	0.7880152382
0.9	0.8913669875	0.8930106719	0.9310084337	0.8928578710	0.8928534411
1	0.9999999999	1.0000000000	1.0427469532	1.0000000000	1.0000000000

TABLE III

Algorithm, variational iteration, Adomian decomposition and B-spline method approximations of Troesch's problem for the case $\lambda = 1.0$.

x	u_{exact}	u_2 [4]	u_{var} [4]	u_{decom} [4]	B-spline
0.1	0.0817969966	0.0854167500	0.1001668336	0.0849252857	0.095943932
0.2	0.1645308709	0.1716693333	0.2013386973	0.1706790842	0.192127937
0.3	0.2491673608	0.2596035833	0.3045410277	0.2581050255	0.288793243
0.4	0.3367322092	0.3500853333	0.4108413284	0.3480781113	0.386183413
0.5	0.4283471610	0.4440104167	0.5213734766	0.4415232987	0.484545552
0.6	0.5252740296	0.5423146667	0.6373663546	0.5394377282	0.584131580
0.7	0.6289711434	0.6459839167	0.7601789657	0.6429180915	0.685199578
0.8	0.7411683782	0.7560640000	0.8913449177	0.7531948954	0.788015238
0.9	0.8639700206	0.8736707500	1.0326302204	0.8716757181	0.892853441
1.0	1.0000000020	1.0000000000	1.1861098656	1.0000000000	1.000000000

TABLE IV

The absolute errors for algorithm, variational iteration, Adomian decomposition and B-spline method approximations of Troesch's problem for the case $\lambda = 0.5$.

x	u_2 [4]	u_{var} [4]	u_{decom} [4]	B-spline
0.1	$8.1789490000 \times 10^{-4}$	$4.8647751000 \times 10^{-3}$	$7.7085210000 \times 10^{-4}$	$7.6703080000 \times 10^{-4}$
0.2	$1.5954642000 \times 10^{-3}$	$9.6997980000 \times 10^{-3}$	$1.5013846000 \times 10^{-3}$	$1.4940686000 \times 10^{-3}$
0.3	$2.2905710000 \times 10^{-3}$	$1.4474136400 \times 10^{-2}$	$2.1500184000 \times 10^{-3}$	$2.1398409000 \times 10^{-3}$
0.4	$2.8574045000 \times 10^{-3}$	$1.9154465800 \times 10^{-2}$	$2.6726236000 \times 10^{-3}$	$2.6604845000 \times 10^{-3}$
0.5	$3.2445156000 \times 10^{-3}$	$2.3703793400 \times 10^{-2}$	$3.0211619000 \times 10^{-3}$	$3.0081667000 \times 10^{-3}$
0.6	$3.3926918000 \times 10^{-3}$	$2.8080020600 \times 10^{-2}$	$3.1422264000 \times 10^{-3}$	$3.1296053000 \times 10^{-3}$
0.7	$3.2326122000 \times 10^{-3}$	$3.2234727100 \times 10^{-2}$	$2.9754375000 \times 10^{-3}$	$2.9644458000 \times 10^{-3}$
0.8	$2.6822133000 \times 10^{-3}$	$3.6110870200 \times 10^{-2}$	$2.4516454000 \times 10^{-3}$	$2.4434515000 \times 10^{-3}$
0.9	$1.6436844000 \times 10^{-3}$	$3.9641446200 \times 10^{-2}$	$1.4908835000 \times 10^{-3}$	$1.4864536000 \times 10^{-3}$
1.0	$1.0000000827 \times 10^{-10}$	$4.2746953300 \times 10^{-2}$	$1.0000000827 \times 10^{-10}$	$1.0000000827 \times 10^{-10}$

TABLE V

The absolute errors for algorithm, variational iteration, Adomian decomposition and B-spline method approximations of Troesch's problem for the case $\lambda = 1.0$.

x	u_2 [4]	u_{var} [4]	u_{decom} [4]	B-spline
0.1	$3.6197534000 \times 10^{-3}$	$1.8369837000 \times 10^{-2}$	$3.1282891000 \times 10^{-3}$	$1.4146935400 \times 10^{-2}$
0.2	$7.1384624000 \times 10^{-3}$	$3.6807826400 \times 10^{-2}$	$6.1482133000 \times 10^{-3}$	$2.7597066100 \times 10^{-2}$
0.3	$1.0436222500 \times 10^{-2}$	$5.5373666900 \times 10^{-2}$	$8.9376647000 \times 10^{-3}$	$3.9625882200 \times 10^{-2}$
0.4	$1.3353124100 \times 10^{-2}$	$7.4109119200 \times 10^{-2}$	$1.1345902100 \times 10^{-2}$	$4.9451203800 \times 10^{-2}$
0.5	$1.5663255700 \times 10^{-2}$	$9.3026315600 \times 10^{-2}$	$1.3176137700 \times 10^{-2}$	$5.6198391000 \times 10^{-2}$
0.6	$1.7040637100 \times 10^{-2}$	$1.1209232500 \times 10^{-1}$	$1.4163698600 \times 10^{-2}$	$5.8857550400 \times 10^{-2}$
0.7	$1.7012773300 \times 10^{-2}$	$1.3120782230 \times 10^{-1}$	$1.3946948100 \times 10^{-2}$	$5.6228434600 \times 10^{-2}$
0.8	$1.4895621800 \times 10^{-2}$	$1.5017653950 \times 10^{-1}$	$1.2026517200 \times 10^{-2}$	$4.6846859800 \times 10^{-2}$
0.9	$9.7007294000 \times 10^{-3}$	$1.6866019980 \times 10^{-1}$	$7.7056975000 \times 10^{-3}$	$2.8883420400 \times 10^{-2}$
1.0	$1.9999999943 \times 10^{-9}$	$1.86109863600 \times 10^{-1}$	$1.9999999434 \times 10^{-9}$	$1.9999999434 \times 10^{-9}$

TABLE VI

Numerical solutions of different values of n for the case $\lambda = 0.5$.

x	u_{exact}	$n = 21$	$n = 61$	$n = 91$
0.1	0.0951769020	0.0959439328	0.09594430303203	0.09594432873001
0.2	0.1906338691	0.1921279377	0.19212865770157	0.19212870767311
0.3	0.2866534030	0.2887932439	0.28879427240158	0.28879434377699
0.4	0.3835229288	0.3861834133	0.38618468720623	0.38618477561355
0.5	0.4815373854	0.4845455521	0.48454698564510	0.48454702512825
0.6	0.5810019749	0.5841315802	0.58413306318382	0.58413316608458
0.7	0.6822351326	0.6851995784	0.68520097399100	0.68520107079903
0.8	0.7855717867	0.7880152382	0.78801638006970	0.78801645923904
0.9	0.8913669875	0.8928534411	0.89285413016780	0.89285417787423
1.0	0.9999999999	1.0000000000	1.00000000015258	$1.9999999434 \times 10^{-9}$