Harmonic Univalent Functions with Janowski Starlike Analytic Part

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Abstract

In this paper we define a new subclass of harmonic univalent functions for which analytic part is Janowski Starlike Function, and investigate some properties of this type of functions. Also we give a new coefficient inequality for harmonic univalent functions.

1 Introduction

Let $\Omega$ be the class of analytic functions $w(z)$ in the open unit disc $\mathbb{D} = \{ z \in \mathbb{C} | |z| < 1 \}$, satisfying $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \mathbb{D}$.

For arbitrary fixed real numbers $A$ and $B$ which satisfy $-1 \leq B < A \leq 1$ we say $p(z)$ belongs to the class $\mathcal{P}(A, B)$ if

$$p(z) = 1 + \sum_{n=1}^{\infty} p_{n}z^{n}$$

is analytic in $\mathbb{D}$ and $p(z)$ is given by

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}$$

for every $z$ in $\mathbb{D}$ and for some $w(z) \in \Omega$. This class, $\mathcal{P}(A, B)$, was first introduced by W. Janowski [3]. Therefore, we call $p(z)$ in the class $\mathcal{P}(A, B)$ "Janowski Function".

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Let $S^*(A, B)$ denote the family of functions

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

regular in $\mathbb{D}$, and such that $h(z)$ is in $S^*(A, B)$ if and only if

$$\frac{h'(z)}{h(z)} = p(z)$$

for some $p(z)$ in $\mathcal{P}(A, B)$ and for every $z \in \mathbb{D}$. Functions in $S^*(A, B)$ are called the “Janowski Starlike Functions” [3].

A continuous complex valued function $f = u + iv$ defined in a simply connected domain $\mathcal{U}$ is said to be “Harmonic” in $\mathcal{U}$ if $u$ and $v$ are real harmonic in $\mathcal{U}$. In any simply connected domain $\mathcal{U} \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where $h$ and $g$ are analytic in $\mathcal{U}$. We call $h$ the “Analytic Part” and $g$ the “Co-Analytic Part” of $f$.

The “Jacobian” of $f$ is given by

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2.$$ 

A necessary and sufficient condition for $f = h + \overline{g}$ is to be locally univalent and sense-preserving in $\mathcal{U}$ such as [2], [4]

$$J_f(z) = |h'(z)|^2 - |g'(z)|^2 > 0.$$ 

This is equivalent to

$$|g'(z)| < |h'(z)|$$

for all $z \in \mathcal{U}$.

Denote by $S_{\mathcal{H}}$ the class of functions $f = h + \overline{g}$ that are “Harmonic Univalent and Sense-Preserving” in the open unit disc $\mathbb{D} = \{z \in \mathbb{C}||z| < 1\}$, for which

$$f(0) = h(0) = f_z(0) - 1 = 0.$$ 

For $f = h + \overline{g} \in S_{\mathcal{H}}$ we may express the analytic functions $h$ and $g$ as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n. \quad (1.1)$$

So, as a result of the sense-preserving property of $f$, $|b_1| < 1$. 
The classical family $S$ which is analytic, univalent and normalized functions on $\mathbb{D}$ is subclass of $S_{\mathcal{H}}$ in which $b_n = 0$ for all $n \in \mathbb{N}$.

The function

$$w_1 = \frac{g'}{h'}$$

is called the "Second Dilatation of $f = h + \bar{g}$", and we denote the class of the second dilatation of $f$ by $\mathcal{W}$. Note that $|w_1(z)| < 1$ and $w_1(0) = b_1 \neq 0$ for all $z$ in $\mathbb{D}$.

We consider the transformation $\phi : \mathbb{C} \rightarrow \mathbb{C}$, given by

$$\phi(z) = \frac{w_1(z) - w_1(0)}{1 - \overline{w_1(0)}w_1(z)}, \quad (1.2)$$

maps the unit disc $\mathbb{D}$ onto itself, where $w_1(z) \in \mathcal{W}$ for every $z$ in $\mathbb{D}$. It is easy to show that $\phi(z)$ is an analytic function in $\mathbb{D}$, and $|\phi(z)| \leq 1$, and $\phi(0) = 0$ for all $z \in \mathbb{D}$. Hence $\phi(z) \in \Omega$.

**Definition 1.1.** Let $f = h + \bar{g} \in S_{\mathcal{H}}$. We define a new subclass of harmonic univalent functions for which analytic part is Janouski starlike function. We denote by $S_{\mathcal{H}}^{*}(A, B)$ the family of all harmonic univalent functions on $\mathbb{D}$ with $h \in S^{*}(A, B)$.

## 2 Auxiliary Lemmas

**Lemma 2.1.** (Schwarz's Lemma [1]) If $\phi(z)$ is analytic for $|z| < 1$ and satisfies the condition $|\phi(z)| \leq 1$, $\phi(0) = 0$ then $|\phi(z)| \leq |z|$ and $|\phi'(0)| \leq 1$. If $|\phi(z)| = z$ for some $z \neq 0$ or if $|\phi'(0)| = 1$, then $\phi(z) = cz$ with a constant $c$ of absolute value $1$.

**Lemma 2.2.** [3] If $h(z) \in S^{*}(A, B)$, then for $|z| = r$, $0 < r < 1$

$$C(r; -A, -B) \leq |h'(z)| \leq C(r; A, B), \quad (2.1)$$

where

$$C(r; A, B) = \begin{cases} 
(1 + Ar)(1 + Br)^{(A-2B)/B}, & \text{if } B \neq 0, \\
(1 + Ar)e^{Ar}, & \text{if } B = 0. 
\end{cases} \quad (2.2)$$

These bounds are sharp, being attained at the point $z = re^{i\varphi}$, $0 \leq \varphi \leq 2\pi$, by

$$h_*(z) = zh_0(z; -A, -B) \quad (2.3)$$
and
\[ h^*(z) = z h_0(z; A, B), \]
respectively, where
\[ h_0(z; A, B) = \begin{cases} (1 + B e^{-i\varphi} z)^{(A-2B)/B}, & \text{for } B \neq 0, \\ e^{-i\varphi} z, & \text{for } B = 0. \end{cases} \]

**Lemma 2.3.** Let \( f = h + \overline{g} \in S_H \) and \( w_1 \in \mathcal{W} \). Then we have
\[ \left| e^{-i\theta} w_1(z) - \frac{\alpha(1-r^2)}{1-\alpha^2r^2} \right| \leq \frac{r(1-\alpha^2)}{1-\alpha^2r^2}, \tag{2.5} \]
where first coefficient of \( g \) is \( b_1 = \alpha e^{i\theta}, 0 \leq \theta \leq 2\pi, \) and \( |z| = r < 1 \). The equality holds in the inequality (2.5) only for the function
\[ w_1(z) = e^{i\beta} \frac{e^{i\theta} z + \alpha}{1 + \alpha e^{i\theta} z}, \quad z \in \mathbb{D}. \tag{2.6} \]

**Proof.** Since \( \phi(z) \) which is given by (1.2) satisfies the conditions of Schwarz’s lemma then \( |\phi(z)| \leq |z| = r < 1 \). Hence, we can write
\[ |\phi(z)| = \frac{|e^{-i\theta} w_1(z) - \alpha|}{|1 - \alpha e^{-i\theta} w_1(z)|} \leq r \Rightarrow |e^{-i\theta} w_1(z) - \alpha| \leq r|1 - \alpha e^{-i\theta} w_1(z)| \]
for all \( z \) in \( \mathbb{D} \). By taking \( e^{-i\theta} w_1(z) = x + iy \) we get following inequality
\[ x^2 + y^2 - 2 \frac{\alpha(1-r^2)}{1-\alpha^2r^2} x + \frac{\alpha^2 - r^2}{1-\alpha^2r^2} \leq 0. \]
So, \( e^{-i\theta} w_1(z) \) maps \( |z| = r \) onto the circle, which has a center of \( C(r) = \left( \frac{\alpha(1-r^2)}{1-\alpha^2r^2}, 0 \right) \) and radius of \( \rho(r) = \frac{r(1-\alpha^2)}{1-\alpha^2r^2} \). \( \qed \)

**Lemma 2.4.** Let \( f = h + \overline{g} \in S_H \) and \( w_1 \in \mathcal{W} \). Then we have
\[ \frac{\alpha - r}{1 - \alpha r} \leq |w_1(z)| \leq \frac{\alpha + r}{1 + \alpha r}, \tag{2.7} \]
for all \( |z| = r < 1 \) and \( |b_1| = \alpha \).

**Proof.** If we use lemma 2.3, we can obtain the result. \( \square \)
3 Main Results

Theorem 3.1. If \( f = h + \bar{g} \in S_{\mathcal{H}} \) be as given in (1.1) and \( w_1 \in \mathcal{W} \), then we have

\[
|b_2| < \frac{1}{2} + |a_2|
\]

for all \( z \) in \( \mathbb{D} \).

Proof. Let's consider the function \( \phi(z) \) which is given by (1.2). Since \( \phi(z) \) satisfies the condition of Schwarz's lemma then \( |\phi'(0)| \leq 1 \). Hence we can write

\[
|\phi'(0)| = \frac{|b_2 - a_2b_1|}{1 - |b_1|^2} < \frac{1}{2}
\] (3.1)

for all \( z \in \mathbb{D} \). By using the definition of the second dilatation function \( w_1(z) \) in (3.1) we get the desired result, after simple calculations.

Lemma 3.2. If \( f = h + \bar{g} \in S_{\mathcal{H}}^*(A, B) \), then we have

\[
C(r; -A, -B) \frac{|\alpha - r|}{1 - \alpha r} \leq |g'(z)| \leq \frac{\alpha + r}{1 + \alpha r} C(r; A, B)
\] (3.2)

where \( C(r; A, B) \) is given by (2.2). The upper and the lower bounds for \( 0 < r < 1 \) are sharp being attained by functions (2.3) and (2.4), respectively.

Proof. Since the definition of the second dilatation function of \( f \) is \( w_1(z) = g'(z)/h'(z) \), then we can write

\[
|g'(z)| = |w_1(z)||h'(z)| \quad (z \in \mathbb{D}).
\] (3.3)

Using (2.1) and (2.7) in (3.3) we obtain desired result.

Theorem 3.3. If \( f = h + \bar{g} \in S_{\mathcal{H}}^*(A, B) \), then for \( |z| = r \), \( 0 < r < 1 \), we have

\[
\int_0^r (1 - A\rho)(1 - B\rho)^{\frac{\alpha - 2\rho}{\beta}} \frac{(1 - \alpha)(1 - \rho)}{(1 + \alpha \rho)} d\rho \leq |f(z)| \leq \int_0^r (1 + A\rho)(1 + B\rho)^{\frac{\alpha + 2\rho}{\beta}} \frac{(1 + \alpha)(1 + \rho)}{(1 + \alpha \rho)} d\rho, \quad \text{for } B \neq 0,
\]

\[
\int_0^r (1 - A\rho)e^{-A\rho} \frac{(1 - \alpha)(1 - \rho)}{(1 + \alpha \rho)} d\rho \leq |f(z)| \leq \int_0^r (1 + A\rho)e^{A\rho} \frac{(1 + \alpha)(1 + \rho)}{(1 + \alpha \rho)} d\rho, \quad \text{for } B = 0,
\]
where $|b_1| = \alpha$ and this bound for $0 < r < 1$ is sharp being attained by functions (2.3), (2.4) and the solution of the differential equation $g'(z) = h'(z)\frac{z+\alpha}{1+\alpha z}$.

Proof. For harmonic univalent function $f = h + \bar{g}$ we know that

$$\int (|h'(z)| - |g'(z)|)|dz| \leq |df(z)| \leq (|h'(z)| + |g'(z)|)|dz|.$$ (3.4)

On the other hand, by using (3.3) we obtain

$$|h'(z)| - |g'(z)| = |h'(z)|(1 - |w_1(z)|)$$ (3.5)

for all $z$ in $\mathbb{D}$. If we use (2.7) and (2.1) in (3.5) we obtain

$$\frac{(1-\alpha)(1-r)}{(1+\alpha r)}C(r; -A, -B) \leq |h'(z)| - |g'(z)|.$$ (3.6)

Furthermore, we have

$$|h'(z)| + |g'(z)| \leq |h'(z)|(1 + |w_1(z)|)$$ (3.7)

for all $z$ in $\mathbb{D}$. Again if we use (2.7) and (2.1) in (3.7) we obtain

$$|h'(z)| + |g'(z)| \leq \frac{(1+\alpha)(1+r)}{(1+\alpha r)}C(r; A, B).$$ (3.8)

By using (3.6) and (3.8) in (3.4) and integrating this inequality form 0 to $r$ we obtain the desired result.

Corollary 3.4. The Heinz's inequality for $f = h + \bar{g} \in S_\mathcal{H}^*(A, B)$ is

$$|h'(z)|^2 + |g'(z)|^2 \geq \begin{cases} (1 - Br)^{2A - 4B} (1 - Ar)^2 \left(1 + \left(\frac{\alpha - r}{1 - \alpha r}\right)^2\right), & B \neq 0, \\ e^{-2Ar}(1 - Ar)^2 \left(1 + \left(\frac{\alpha - r}{1 - \alpha r}\right)^2\right), & B = 0, \end{cases}$$

for all $z \in \mathbb{D}$, and $|b_1| = \alpha$.

Proof. Since $g'(z) = w_1(z)h'(z)$ for all $z \in \mathbb{D}$, then

$$|h'(z)|^2 + |g'(z)|^2 = |h'(z)|^2(1 + |w_1(z)|^2).$$ (3.9)

If we use the inequalities (2.1) and (2.7) in (3.9) we get the result, after simple calculations.
Theorem 3.5. If \( f = h + \bar{g} \in S_{H}^{*}(A, B) \), then
\[
C^{2}(r; -A, -B) \frac{(1 - r^{2})(1 - \alpha^{2})}{(1 + \alpha r)^{2}} \leq J_{f}(z) \leq C^{2}(r; A, B) \left(1 - \frac{|\alpha - r|^{2}}{(1 - \alpha r)^{2}}\right)
\]
for all \( z \in \mathbb{D} \), and \( |b_{1}| = \alpha \).

Proof. Using lemma 2.4 and the relations
\[
J_{f}(z) = |h'(z)|^{2} - |g'(z)|^{2}
\]
and
\[
g'(z) = w(z)h'(z)
\]
we obtain the result. \( \square \)

Note. If we consider the spacial values for \( A \) and \( B \) as below, we can obtain some subclasses.

- \( A = 1, B = -1 \).
- \( A = 1 - 2\alpha \ (0 \leq \alpha < 1), B = -1 \).
- \( A = 1, B = \frac{1}{M} - 1 \ (M > \frac{1}{2}) \).
- \( A = \beta, B = -\beta \ (0 \leq \beta < 1) \).

References


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