An Investigation on the Janowski Convex Function of Complex Order

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Abstract
In this paper We shall give some results for the Janowski convex function of complex order.

Özet
Bu makalede kompleks mertebeden Janowski konveks fonksiyonları için bazı neticeler veririz.

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I. Introduction
Let \( \Omega \) be the family of functions \( \Psi(z) \) regular in the unit disc \( D = \{ ... | \ | z | < 1 \} \), and satisfying the conditions \( \Psi(0) = 0, \ | \ \Psi(z) | < 1 \) for \( z \in D \).

Next, for arbitrary fixed numbers \( A, B, -1 \leq B < A \leq 1 \) denote by \( P(A,B) \) the family of functions \( p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... \) regular in \( D \) such that is in \( P(A,B) \) if and only if

\[
p(z) = \frac{1 + A \Psi(z)}{1 + B \Psi(z)}
\]

for some function \( \Psi(z) \in \Omega \) and every \( z \in D \).

\[
f(z) = z + a_2 z^2 + a_3 z^3 + ... , f'(z) \neq 0
\]

regular in \( D \) and such that \( f(z) \) is in \( C(A,B,b) \) if and only if

\[
1 + \frac{1}{b z} \frac{f''(z)}{f'(z)} = p(z)
\]

for some \( p(z) \in P(A,B) \) and all \( z \in D \).

Finally, we consider the following class of functions which are regular in \( D \) and

\[
f(z) = z + a_2 z^2 + a_3 z^3 + ... , \frac{f(z)}{z} \neq 0, 1 + \frac{1}{b} (z, \frac{f'(z)}{f(z)} - 1) = p(z) \text{ for some }
\]

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\( p(z) \in P(A,B) \) and for all \( z \) in \( D \). This class is denoted by \( S^*(A,B,b) \).

We note that \( P(1,-1) \) is the class of Caratheodory functions. Therefore \( C(A,B,b) \) contains the following classes.

1) \( C(1,-1,1) \) is the well known class of convex functions [1].
2) \( C(1,-1,b) \) is the class of convex functions of complex order. This class was introduced by P. Wiatrowski [8], and M. K. Nasr and M. K. Aouf [3], [4].
3) \( C(1,-1,-\beta), 0 \leq \beta < 1 \) is the class of convex functions of order \( \beta \). This class was introduced by M. S. Robertson [5].
4) \( C(1,-1, e^{-i\lambda} \cos \lambda), |\lambda| < \frac{\pi}{2} \) is the class of functions for which \( z.f'(z) \) is \( \lambda \)-Spirallike functions. This class was introduced by M. S. Robertson [6].
5) \( C(1,-1, (1-\beta).e^{-i\lambda} \cos \lambda), 0 \leq \beta < 1, |\lambda| < \frac{\pi}{2} \) is the class of functions for which \( z.f'(z) \) is \( \lambda \)-Spirallike of order \( \beta \). This class was introduced by P. I. Sizuk [7].

If we write \( CT(b) = 1 + \frac{1}{b}.z.f'''(z) \)

6) \( C(1,0,b) \) is the set defined by \( |CT(b) - 1| < 1 \)
7) \( C(\beta,0,b), 0 \leq \beta < 1 \) is the set defined by \( |CT(b) - 1| < \beta \)
8) \( C(\beta,-\beta,b), 0 \leq \beta < 1 \) is the set defined by \( \frac{|CT(b) - 1|}{|CT(b) + 1|} < \beta \)
9) \( C(1,(-1+\frac{1}{M}),b), M \geq 1 \) is the set defined by \( |CT(b) - M| < M \)
10) \( C(1-2\beta,-1,b), 0 \leq \beta < 1 \) is the set defined by \( \text{Re} CT(b) \geq \beta \)

II. Auxiliary Lemmas.

In this section of this paper we shall give the following lemmas for the purpose of this paper.

**Lemma 2.1** Let \( f(z) \in C(A,B,b) \), then the function \( g(z) = z.f'(z) \) belongs to \( S^*(A,B,b) \).

**Proof.** If we take the logarithmic derivative from the equality \( g(z) = z.f'(z) \) we get
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(2.1) \[ z \frac{g'(z)}{g(z)} = 1 + z \frac{f'''(z)}{f'(z)}. \]

After the simple calculations from the equality (2.1) we obtain

(2.2) \[ 1 + \frac{1}{b} \left( z \frac{g'(z)}{g(z)} - 1 \right) = 1 + \frac{1}{b} z \frac{f'''(z)}{f'(z)}. \]

The equality (2.2) shows that this lemma is true.

**Lemma 2.2.** Let \( f(z) \in C(A,B,b) \) then the derivative of \( f(z) \) is given by the relation

\[
f'(z) = \begin{cases} 
(1 + B\Psi(z))^{\frac{b(A-B)}{b}} & \text{if } B \neq 0 \\
e^{b\Psi(z)} & \text{if } B = 0
\end{cases}
\]

Where \( \Psi(z) \in \Omega \)

**Proof:**

Step one: Let \( B \neq 0 \) and \( f'(z) = (1 + B\Psi(z))^{\frac{b(A-B)}{b}} \).

If we take the logarithmic derivative from this equality and a simple calculations we obtain

(2.3) \[ z \frac{f''(z)}{f'(z)} = b(A-B) \frac{z\Psi'(z)}{\Psi(z)}. \]

By using I.S.Jack's Lemma [2] in the equality (2.3) and a simple calculations shows that
\[ (2.4) \quad 1 + \frac{1}{b} z \frac{f''(z)}{f'(z)} = \frac{1 + A\Psi(z)}{1 + B\Psi(z)}. \]

The equality (2.4) shows that \( f(z) \in C(A, B, b) \).

Step two: Let \( B = 0 \), and \( f'(z) = e^{h(A-B)} \). Similarly we obtain

\[ (2.5) \quad 1 + \frac{1}{b} z \frac{f''(z)}{f'(z)} = \frac{1 + A\Psi(z)}{1 + B\Psi(z)} \]

the equality (2.5) shows that \( f(z) \in C(A, B, b) \)

**Corollary 2.1.** From the Lemma 2.1 and Lemma 2.2 we obtain that the function

\[
 f(z) = \begin{cases} 
 z^{b(A-B)/B} & B \neq 0 \\
 \int_0^z e^{h\zeta} d\zeta & B = 0
\end{cases}
\]

belongs to \( C(A, B, b) \)

**Theorem 2.1** The set \( C(A, B, b) \) is invariant under the rotation, so that \( e^{-ia} f(e^{ia} z) \) is in \( C(A, B, b) \) whenever \( f(z) \) is in \( C(A, B, b) \)

**Proof:** Let \( f(z) \) is in \( C(A, B, b) \) then the equality

\[ (2.6) \quad 1 + \frac{1}{b} z \frac{f''(z)}{f'(z)} = \frac{1 + A\Psi(\zeta)}{1 + B\Psi(\zeta)} \]

is satisfied. Where \( \Psi(\zeta) \in \Omega \). On the other hand if we write

\[ (*) \quad g(z) = e^{-ia} f(e^{-ia} z), \]

and we take the logarithmic derivative from the relation (*) and a simple calculations shows that
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\begin{equation}
(2.7) \quad 1 + \frac{1}{b} \cdot z \cdot \frac{g''(z)}{g'(z)} = 1 + \frac{1}{b} \cdot e^{i\alpha} \cdot z \cdot \frac{f''(e^{i\alpha} z)}{f'(e^{i\alpha} z)}.
\end{equation}

Now if we take $\zeta = e^{i\alpha} \cdot z$ then we have

\begin{equation}
(2.8) \quad |\zeta| = |e^{i\alpha} \cdot z| = |e^{i\alpha}| \cdot |z| \leq 1 \cdot |z| < 1
\end{equation}

Consider the relations (2.6), (2.7) and (2.8) together we obtain that

\begin{equation}
(2.9) \quad 1 + \frac{1}{b} \cdot z \cdot \frac{g''(z)}{g'(z)} = \frac{1 + A\Psi(z)}{1 + B\Psi(z)}.
\end{equation}

References

7. P.I. Sizuk., (1975), "Regular functions $f(z)$ for which $f'(z)$ is $\theta$-shaped of order $\alpha$", Sibirsk. Math. Z. 16. 1286-1289.