B-spline solution of linear hyperbolic partial differential equations

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Abstract— *Second-order linear hyperbolic equations are solved by using B-spline method . The numerical solution of the equations are discussed and illustrated with an example. Numerical results reveal that B-spline method is implemented and effective.*

Keywords: Second-order linear hyperbolic equations;Finite difference;B-spline functions;Boundary conditions.

1. Introduction

We consider the second-order linear hyperbolic equation:

$$u_{tt}(x,t) + 2\alpha u_t(x,t) + \beta^2 u(x,t) = u_{xx}(x,t) + f(x,t), \quad x \in (a,b), \quad t > 0 \tag{1}$$

with initial conditions $u(x,0) = \Phi(x), \quad u_t(x,0) = \Psi(x)$ and boundary conditions $u(a,t) = g_1(t), \quad u(b,0) = g_2(t)$, where α and β are constants.

The equation above represents a damped wave equation and a telegraph equation, the existence and approximations of the solutions investigated in literature. In recent years, many research has been done in developing and implementing modern high resolutions methods for the numerical solution of the second-order linear hyperbolic equation(1).In recent years, many research has been done in developing and implementing modern high resolutions methods for the numerical solution of the second-order linear hyperbolic equation(1), see$[8-15]$. Recently, Gao and Chi$[8]$ proposed two semi-discretion methods to solve the one-space dimensional linear hyperbolic equation(1). Also, Huan-Wen Liu and Li-Bin Liu solved$[8]$ linear hyperbolic equation. In this paper, we propose a B-spline difference scheme to solve the linear hyperbolic equation(1).

The present paper will focus on a new method of solution of the linear hyperbolic equation by using third degree B-spline functions. The theory of spline functions is a very active field of approximation theory and boundary value problems (BVPs), when numerical aspects are considered. In a series of paper by Caglar et al. [2-7] BVPs of order two, third, fourth and fifth were solved using third, fourth and sixth-degree splines.

We propose B- spline difference scheme to solve the linear hyperbolic equation(1). The numerical results obtained by using the method described in this study give acceptable results. We have concluded that numerical results converge to the exact solution when k goes to zero and for smaller h the maximum absolute error decreased. In this paper , we have derived a new method based on B- splines for solution (1). In Section 2 , we give a brief derivation of B-spline function. In Section 3, the method are used to analysis to solution of problem (1). In Section 4, some numerical result, that are illustrated using MATLAB 6.5, are given to clarify the method. Finally, in Section 5 ends this paper with a brief conclusion.

2. The third-degree B-splines

In this section, third-degree B-splines are used to construct numerical solutions to the hyperbolic equations discussed in sections 3 and 4. A detailed description of B-spline functions generated by subdivision can be found in [1]. Consider equally-spaced knots of a partition $\pi : a = x_0 < x_1 < ... < x_n = b$ on [a,b]. Let $S_3[\pi]$ be the space of continuously-differentiable, piecewise, third-degree polynomials on π. That is, $S_3[\pi]$ is the space of third-degree splines on π. Consider the B-splines basis in $S_3[\pi]$. The third-degree B-splines are defined as

$$B_0(x) = \frac{1}{6h^3}\begin{cases} x^3 & 0 \le x < h \\ -3x^3 + 12hx^2 - 12h^2x + 4h^3 & h \le x < 2h \\ 3x^3 - 24hx^2 + 60h^2x - 44h^3 & 2h \le x < 3h \\ -x^3 + 12hx^2 - 48h^2x + 64h^3 & 3h \le x < 4h \end{cases}$$

$$B_{i-1}(x) = B_0(x-(i-1)h), \ i = 2, 3, ...,$$

To solve hyperbolic equation, B_i , B_i' *and* B_i'' evaluated at the nodal points are needed. Their coefficients are summarized in Table 1.

Table 1
VALUES OF B_i , B_i' *and* B_i''

	$\mathbf{x}_i$	$\mathbf{x}_{i+1}$	$\mathbf{x}_{i+2}$	$\mathbf{x}_{i+3}$	$\mathbf{x}_{i+4}$
B_i	0	1/6	4/6	1/6	0
B_i'	0	$3/6h$	$0/6h$	$-3/6h$	0
B_i''	0	$6/6h^2$	$-12/6h^2$	$6/6h^2$	0

3. B-spline solutions for hyperbolic equation

In this section the B-spline method for solving hyperbolic equation is outlined, which is based on the collocation approach[8]. We seek a function $S(x)$ that approximates the solution of hyperbolic equation(1), may be represented as

$$S(x) = \sum_{j=-3}^{n-1} C_j B_j(x), \quad (3)$$

where C_i are unknown real coefficients and $B_j(x)$ are third-degree B-spline functions. Let $x_0, x_1, ..., x_n$ be $n+1$ grid points in the interval $[a, b]$, so that

$$x_i = a + ih, i = 0, 1, ..., n; x_{0=}a, x_n = b, h = (b-a)/n.$$

We consider the equation (1),

difference schemes for this problem considered as following:

$$\frac{u_{i+1}-2u_i+u_{i-1}}{\Delta t^2} + 2\alpha \frac{u_i - u_{i-1}}{\Delta t} + \beta^2 u = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad (4)$$

where $\Delta t = k$

$$-u''_{i+1} + (\tfrac{1}{k^2} + \beta^2)u_{i+1} = (\tfrac{2}{k^2} - \tfrac{2\alpha}{k})u_i + (\tfrac{2\alpha}{k} - \tfrac{1}{k^2})u_{i-1} + f(x,t), \quad (5)$$

and the initial conditions are given in (8)-(9)

$$u(x, 0) = \phi(x) = u_0, u(k, x) = u_1, \quad (6)$$

$$u_t(x, 0) = \psi(x) = (u_1 - u_0)/k, \quad (7)$$

$$u_1 = u_0 + k\psi(x). \quad (8)$$

Subsituting (6-8) in (5) then is obtained as follows

$$t = 2k$$
$$-u''_2 + (\tfrac{1}{k^2} + \beta^2)u_2 = (\tfrac{2}{k^2} - \tfrac{2\alpha}{k})u_1 + (\tfrac{2\alpha}{k} - \tfrac{1}{k^2})u_0 + f(x,t), \quad (9)$$

$$t = 3k$$
$$-u''_3 + (\tfrac{1}{k^2} + \beta^2)u_3 = (\tfrac{2}{k^2} - \tfrac{2\alpha}{k})u_2 + (\tfrac{2\alpha}{k} - \tfrac{1}{k^2})u_1 + f(x,t), \quad (10)$$

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$$t = nk \quad -u''_n + (\tfrac{1}{k^2} + \beta^2)u_n = (\tfrac{2}{k^2} - \tfrac{2\alpha}{k})u_{n-1} + (\tfrac{2\alpha}{k} - \tfrac{1}{k^2})u_{n-2} + f(x,t), \quad (11)$$

The approximate solution of the equation (9)-(11) are sought in the form of the B-spline functions $S(x)$, it follows that

$$t = 2k$$
$$-S''_2 + (\tfrac{1}{k^2} + \beta^2)S_2 = (\tfrac{2}{k^2} - \tfrac{2\alpha}{k})u_1 + (\tfrac{2\alpha}{k} - \tfrac{1}{k^2})u_0 + f(x,t), \quad (12)$$

$$t = 3k$$
$$-S''_3 + (\tfrac{1}{k^2} + \beta^2)S_3 = (\tfrac{2}{k^2} - \tfrac{2\alpha}{k})u_2 + (\tfrac{2\alpha}{k} - \tfrac{1}{k^2})u_1 + f(x,t), \quad (13)$$

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$$t = nk \quad -S''_n + (\tfrac{1}{k^2} + \beta^2)S_n = (\tfrac{2}{k^2} - \tfrac{2\alpha}{k})u_{n-1} + (\tfrac{2\alpha}{k} - \tfrac{1}{k^2})u_{n-2} + f(x,t), \quad (14)$$

and boundary conditions

$$\sum_{j=-3}^{n-1} C_j B_j(x) = g_1(t) \text{ for } x = 0, \quad (15)$$

$$\sum_{j=-3}^{n-1} C_j B_j(x) = g_2(t) \text{ for } x = 1, \quad (16)$$

The spline solution of eq.(12) with the boundary conditions are obtained by solving to the following matrix equation. The value of spline functions at the knots $\{x_i\}_{i=0}^n$ are determined using Table 1. Then the B-spline method in matrix form can be written as follows

$$AC = F$$

where

$$C = [\ C_{-3}\ , C_{-2}\ , C_{-1}\ , \ldots , C_{n-3}\ , C_{n-2}\ , C_{n-1}\ ]^T,$$

$$F = \left[\begin{array}{c} g_1(2k) \\ (\tfrac{2}{k^2} - \tfrac{2\alpha}{k})u_1(x_0) + (\tfrac{2\alpha}{k} - \tfrac{1}{k^2})u_0(x_0) + f(2k, x_0) \\ (\tfrac{2}{k^2} - \tfrac{2\alpha}{k})u_1(x_1) + (\tfrac{2\alpha}{k} - \tfrac{1}{k^2})u_0(x_1) + f(2k, x_1) \\ \vdots \\ (\tfrac{2}{k^2} - \tfrac{2\alpha}{k})u_1(x_{n-1}) + (\tfrac{2\alpha}{k} - \tfrac{1}{k^2})u_0(x_{n-1}) + f(2k, x_{n-1}) \\ (\tfrac{2}{k^2} - \tfrac{2\alpha}{k})u_1(x_n) + (\tfrac{2\alpha}{k} - \tfrac{1}{k^2})u_0(x_n) + f(2k, x_n) \\ g_2(2k) \end{array} \right.$$

Also the matrix A can be writen as

$$A = \begin{bmatrix} \frac{1}{6} & \frac{4}{6} & \frac{1}{6} & 0 & 0 & \dots & 0 \\ \varphi_1 & \varphi_2 & \varphi_3 & 0 & 0 & \dots & 0 \\ 0 & \varphi_1 & \varphi_2 & \varphi_3 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & \dots & \varphi_1 & \varphi_2 & \varphi_3 & 0 \\ 0 & 0 & 0 & \dots & \varphi_1 & \varphi_2 & \varphi_3 \\ 0 & 0 & 0 & \dots & \frac{1}{6} & \frac{4}{6} & \frac{1}{6} \end{bmatrix},$$

where

$$\varphi_1 = -\frac{1}{h^2} + (\frac{1}{k^2} + \beta^2)\frac{1}{6},$$

$$\varphi_2 = \frac{12}{6h^2} + (\frac{1}{k^2} + \beta^2)\frac{4}{6},$$

$$\varphi_3 = -\frac{6}{6h^2} + (\frac{1}{k^2} + \beta^2)\frac{1}{6}.$$

It is easy to see that, the same approximation can be applied the other equations (13)-(14).

4. Numerical results

In this section, the method discussed in section 2 and 3 is tested on the following problem from the literature[9], and the maximum absolute errors in the analytical solutions are calculated. Also we compare our results with Liu et all[4] and Mahonty[14] in Table 3-4. Our methods has its own advantages, once the solution has been simple algorithm and computational. All computations were carried out using MATLAB 6.5.

Example: We consider the following equation[9]

$$u_{tt}(x,t) + 2u_t(x,t) + \beta^2 u(x,t) = u_{xx}(x,t) + (4 - 4\alpha + \beta^2 + h^2)e^{-2t} sinhx,$$

$$\alpha > \beta \geq 0 \ , x \in (a,b), \quad t > 0$$

with initial conditions

$$u(x,0) = sinhx, \quad u_t(x,0) = -2sinhx$$

and boundary conditions

$$u(0,t) = 0, \quad u(1,0) = e^{-2t} sinh$$

Table 2
ABSOLUTE ERRORS OF B-SPLINE SOLUTION

h	$k = 0.01$	$k = 0.001$	$k = 0.0001$
$\frac{1}{16}$	9.6419e-04	9.8278e-05	9.8466e-06
$\frac{1}{32}$	4.8150e-04	4.9062e-05	4.9155e-06
$\frac{1}{64}$	2.4035e-04	2.4490e-05	2.4536e-06
$\frac{1}{121}$	1.2710e-04	1.2951e-05	1.2976e-06
$\frac{1}{521}$	2.9515e-05	3.0074e-06	3.0131e-07

Table 3
ABSOLUTE ERRORS OF B-SPLINE SOLUTION AND COMPARE WITH THE FINITE DIFFERENCE SCHEME

h	$t = 1$	$t = 2$	$t = 1$	t=2
	finite difference		B-spline	
$\frac{1}{16}$	0.6386e-02	0.5937e-02	0.95892e-03	0.75376e-03
$\frac{1}{32}$	0.2229e-02	0.1800e-02	0.48085e-03	0.37565e-03
$\frac{1}{64}$	0.6002e-03	0.4826e-03	0.24004e-03	0.18798e-03

The exact solution of the above problem is $u(x,t) = e^{-2t} sinhx$. The observed maximum absolute errors for different values of step size h and k are given in Table 2 . for $= 50, \beta = 5$. Also numerical results are shown in Fig. 1. The maximum absolute errors at t=1,2 for h=1/16,1/32,1/64 are tabulated in tables 3-4.

5. Conclusions

In this paper, a family of B-spline methods has been considered for the numerical solution of the hyperbolic equation. The third-degree B-spline was tested on hyperbolic equation and the maximum absolute errors have tabulated. The results showed that the present method is an applicable technique and approximates the exact solution very well.

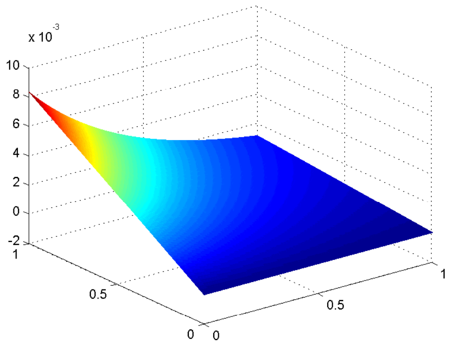

Fig. 1
RESULTS FOR $n = 121, k = 0.0001$.

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