Journal of İstanbul Kültür University
2002/1, pp. 67- 74.

Distortion Theorem and Koebe Domain for Starlike Functions of Complex Order

Yaşar Polatoğlu *, Metin Bolcal*

Abstract
The aim of this paper is to give distortion theorem and Koebe domain for the class of starlike functions of complex order under the conditions.

$$|b| \le \frac{\mathrm{Re}\left(\frac{1}{b}\right) - |z|\mathrm{Re}\left(\frac{1}{b}\right) - 2|z|^2}{2\left|z\left[\mathrm{Re}\left(\frac{1}{b}\right) + \left(2 - \mathrm{Re}\left(\frac{1}{b}\right)\right)\right]\right|}, \quad \mathrm{Re}\left(\frac{1}{b} - 1\right) > 0, |z| < 1$$

Özet
Bu çalışmanın amacı

$$|b| \le \frac{\mathrm{Re}\left(\frac{1}{b}\right) - |z|\mathrm{Re}\left(\frac{1}{b}\right) - 2|z|^2}{2\left|z\left[\mathrm{Re}\left(\frac{1}{b}\right) + \left(2 - \mathrm{Re}\left(\frac{1}{b}\right)\right)\right]\right|}, \quad \mathrm{Re}\left(\frac{1}{b} - 1\right) > 0, |z| < 1$$

koşulu altında kompleks mertebeden yıldızıl fonksiyonlar için yeni bir distorsiyon teoremi ve Koebe bölgesini vermektir.
Keywords: Starlike functions of complex order, Distortion theorem, Koebe domain, Carattheodory functions.

Introduction:

Let A denote the class of functions

$$f(z) = z + a_2 z^2 + \ldots\ldots\ldots$$

Which are analytic in $D = \{z \mid |z| < 1\}$

Definition I.I.

Let $f(z) \in A$ then f(z) is said to be starlike functions of complex order b (b ≠ 0, complex), that is $f(z) \in S(1-b)$ if and only if $\frac{f(z)}{z} \neq 0$ in D and

$$\mathrm{Re}\left[1 + \frac{1}{b} z\left(\frac{f'(z)}{f(z)} - 1\right)\right] > 0$$

---

* Department of Mathematcs, İstanbul Kültür Üniversity Şirinevler 34510 İstanbul-Türkiye

We note that

(i) $S(0)=S^{*}$ is the. Class of starlike functions (well known class [see 1 volume I and 2 and see 2]

(ii) $S(\beta)=S^{\beta}$, $b\leq \beta< 1$ is the class of starlike functions of order β. This class was introduced by M.S.Robertson [4]

(iii) $S(1-e^{-i\lambda}Cos\lambda)$, $|\lambda|< \frac{\pi}{2}$ is the class of λ -spirallike functions introduced by L. Spacek [see 1. Volume I and II and see 2]

(iv) $S(1-(1-\beta)e^{-i\lambda}Cos\lambda)$, $0\leq \beta< 1$, $|\lambda|< \frac{\pi}{2}$ is the class of λ -spirallike of order β. This class was introduced by Libera [3]

Definition 1.2.

Let $f(z)\in A$, then f(z) is said to be convex functions of complex order b ($b\neq 0$, complex) that is $f(z)\in C(b)$ if and only if $f'(z)\neq 0$ in D and

$$\mathrm{Re}\left[1+\frac{1}{b}z\left(\frac{f''(z)}{f'(z)}\right)\right]>0$$

Definition 1.3

Let A be a set of functions f(z); each regular in D. The Koebe domain for a set A is denoted by K(A) and is the collection of points w such that w is in f(D) for every function f(z) in A. In symbols

$$K(A)=\bigcap_{f\in A} f(D)$$

Supposing that the set A is invariant under the rotation, so $e^{-i\alpha}f(e^{i\alpha}z)$ is in A whenever f(z) in A. Then the Koebe domain willl be either the single point w=0 or an open disk $|w|< R$. In the second case R is often easy to find. Indeed supposing that we have a sharp lower bound $M(r)$ for $f(re^{i\theta})$ for all functions in A, and A contains only univalent functions then

$$R=\lim_{r\to 1^-} M(r)$$

Gives the disc $|w|< R$ as the Koebe domain for the set A.

Definition 1.4.

Let $f(z) = 1 + p_1 z + p_2z^2 + \dots\dots\dots$ be analytic in D and satisfies the conditions $\mathrm{Re}\, p(z) > 0$,$p(0) =1$, then this function is called Caratheodory function, then the class of canatheodory functions is denoted by P.

Theorem.1.1.
Let $p(z)\in P$ then

$$(1,1)\qquad \left|\frac{zp'(z)}{p(z)+\beta}\right|\leq\frac{z|z|}{(1+|z|)((1+\beta)+(1-\beta)|z|)}$$

$$(1.2)\qquad |p(z)-1|<\frac{2|z|}{(1-|z|)}$$

This theorem was proved by S.D.Bernardi [5] and M.S.Robertson [4]

Theorem.1.2.
Let g(z) be analytic in D and $g'(z)\neq 0$. If

$$(1.3)\qquad \left(1-|z|^2\right)\left|z\frac{g''(z)}{g(z)}\right|<1 \quad ,z\in D$$

Then g(z) is univalent in D. This theorem was proved by Duren and Shapiro [6].

Distortion and Koebe Domain for Starlike Functions of Complex Order.
In this section of this paper we shall give a new distortion theorem and Koebe domain for the class S(1-b)

Lemma.2.1.
Let $f(z)\in S(1\text{-}b)$, then a sufficient condition for the univalence of f(z) is

$$|b|\leq\frac{\operatorname{Re}\left(\frac{1}{b}\right)-|z|\operatorname{Re}\left(\frac{1}{b}\right)-2|z|^2}{2|z|\left[\operatorname{Re}\left(\frac{1}{b}\right)+\left(2-\operatorname{Re}\left(\frac{1}{b}\right)\right)|z|^2\right]} \quad , \operatorname{Re}\left(\frac{1}{b}-1\right)>0\ ,z\in D$$

Proof: From the definition 1.1 and 1.4 we write

$$(2.1)\qquad z\left(\frac{f'(z)}{f(z)}\right)=b(p(z)-1)+1 \quad ,z\in D$$

If we take the logarithmic derivative from the (2.1) we obtain.

$$(2.2)\qquad z\left(\frac{f''(z)}{f'(z)}\right)=b(p(z)-1)+\frac{zp'(z)}{p(z)+\left(\frac{1}{b}-1\right)}$$

Now we consider the relations (1.1), (1.2), (1.3) and (2.2) together, then we see that this lemma is true

Lemma.2.2

Let f(z) be rgular in unit circle and normalized so that $f(0)=f'(0)-1=0$
A necessary and sufficient condition for $f(z) \in C(b)$ is that for each member $s(z) \in S(1\text{-}b)$ the equation.

$$(2.3) \qquad s(z) = z\left(\frac{f(z)-f(\eta)}{z-\eta}\right)^2 \qquad z,\eta \in D\ ,\ z \neq \eta$$

must be satisfied.

Proof:
Let f(z) convex function of complex order in D, then the function s(z) which is defined by the relation (2.3) is analytic, regular and continuous in the unit disc. Therefore, by using continuity, the equation (2.3) can be written in the form:

$$(2.4) \qquad s(z) = z\left(f'(z)\right)^2$$

$$(2.5) \qquad \mathrm{Re}\left[\frac{1}{2b}\left(z\frac{s'(z)}{s(z)}-1\right)+1\right] = \mathrm{Re}\left[1+\frac{1}{b}z\frac{f''(z)}{f'(z)}\right]$$

Considering the relation (2.5) and the definitions of convex functions complex order, and the definition of starlike function of complex order together we conclude the function s(z) is starlike functions of complex order.

Conversely:
Let s(z) is starlike functions of complex order in D, then simple calculations form (2.3) we obtain that

$$(2.6) \qquad \left[\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)+1\right] = \frac{1}{b}\left[\frac{2zf'(z)}{f(z)-f(\eta)}-\frac{z+\eta}{z-\eta}\right]+\frac{b-1}{b}$$

If we write

$$F(z,\eta) = \frac{1}{b}\left[\frac{2zf'(z)}{f(z)-f(\eta(}-\frac{z+\eta}{z-\eta}\right]+\frac{b-1}{b}$$

The relation (2.6) can be written in the form

$$(2.7) \qquad F(z,\eta) = \left[\frac{1}{b}\left(z\frac{s'(z)}{s(z)}-1\right)+1\right]$$

Considering the relation (2.7) and the definition of starlike function of complex order together we obtain

(2.8) $$\operatorname{Re} F(z,\eta) > 0$$

(2.9) $$F(z,\eta) = 1 + \frac{1}{b}\left(\frac{2}{f(\eta)} - \frac{2}{\eta}\right)z + \ldots\ldots\ldots.$$

(2.10) $$\lim_{\eta \to z} F(z,\eta) = \left[1 + \frac{1}{b} z \frac{f^{//}(z)}{f^{/}(z)}\right]$$

Therefore, by using continutiy, the claim is proved. Hence it follows that f(z) is convex function of complex order.

Theorem. 2.1.

Let f(z) = S(1-b).Then

(2.11) $$\frac{2r}{(1+|b|)(1+r)^2} < |f(z)| < \frac{2r}{(1+|b|)(1-r)^2}$$

The limits are attained by the function

$$f_*(z) = \frac{z}{(1+|b|)(1+z)^2}$$

Proof:

Let $h(z) \in C(b)$, then from lemma (2.2), the function

$$F(z,\eta) = \frac{1}{b}\left[\frac{2zh^{/}(z)}{h(z) - h(\eta(} - \frac{z+\eta}{z-\eta}\right] + \frac{b-1}{b} = 1 + \frac{1}{b}\left(\frac{2}{h(\eta)} - \frac{2}{\eta}\right)z + \ldots\ldots\ldots$$

Belongs the class P. Therefore from the charatheodory inequality we can write

$$\left|\frac{1}{b}\left(\frac{2}{h(\eta)} - \frac{2}{\eta}\right)\right| < 2$$

The last inequality can be written in the form

(2.12) $$\operatorname{Re}\left(\frac{1+|b|}{2} \cdot \frac{h(z)}{z}\right) > \frac{1}{2}$$

Therefore, the function

$$\left(\frac{1+|b|}{2}.\frac{h(z)}{z}\right)$$

is subordinate to the function, $\left(\frac{1}{1-z}\right)$

using the subordination principle we can write

$$\frac{1+|b|}{2}h(z)=\frac{z}{1-\varphi(z)} \tag{2.13}$$

Where $\varphi(z)$ is analytic in D and satisfies the conditions $\varphi(0)=0$ $|\varphi(z)|<1$.

If we take differentiating from (2.13) we obtain

$$\frac{1+|b|}{2}h'(z)=\frac{1+z\varphi'(z)-\varphi(z)}{(1-\varphi(z))^2} \tag{2.14}$$

If we use Jack's Lemma [3]. In (2.14) we find that

$$\frac{1+|b|}{2}zh'(z)=\frac{z}{(1-\varphi(z))^2} \tag{2.15}$$

It is clear that the relation between the classes S(1-b) and C(b) is

$$h(z)\in C(b) \Leftrightarrow zh'(z)=f(z)\in S(1-b) \tag{2.16}$$

therefore, the inequality (2.15) can be written in the form

$$\frac{1+|b|}{2}f(z)=\frac{z}{(1-\varphi(z))^2} \tag{2.17}$$

where $f(z)\in S(1-b)$.(2.17) shows that the function

$$\left[\frac{1+|b|}{2}\frac{f(z)}{z}\right]$$

is subordinate to the Koebe function

$$\left[\frac{z}{(1-z)^2}\right]$$

Finally using the subordination principle, we obtain (2.14). This is a new distortion theorem for the class S81-b)

Corollary 2.1

The following special cases are obtained by giving special values to b.

(i) b = 1 , $$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}$$

This is a well-known result which is distortion theorem of starlike functions [1]

Corollary 2.2.

If we take the limit from $r \to 1$ and using the definition (1.3) we obtain the Keobe domain for the class S(1-b), which is

$$R = \frac{1}{2(1+|b|)}$$

If we give special values to b, we obtain the following results:

(i) b=1 , $R = \frac{1}{4}$ is a well-known result. This is the Keobe domain for the class of starlike functions

(ii) $b = 1 - \alpha$, $$R = \frac{1}{2(2-\alpha)}$$

This result is Keobe domain for the class of starlike functions of order $\alpha.(0 \le \alpha < 1)$

(iii) $b = (1-\alpha)e^{-i\lambda}\mathrm{Cos}\lambda$ $(0 \le \alpha < 1)$, $|\lambda| < \frac{\pi}{2}$

$$R = \frac{1}{2\left[1+(1-\alpha)\mathrm{Cos}\lambda\right]}$$

This is the Keobe domain for the class of spirallike functions of order α.

(iv) $b = e^{-i\lambda}\mathrm{Cos}\lambda$, $|\lambda| < \frac{\pi}{2}$

$$R = \frac{1}{2(1+\mathrm{Cos}\lambda)}$$

this is the Keobe domain of starlike functions of spirallike functions.

Remark.

Robertson ([1]) proved the Keobe domain of starlike functions of order α to be

$$R_1 = \frac{1}{4^{1-\alpha}}$$

and we find that the Keobe domain for the same class is

$$R_2 = \frac{1}{2(2-\alpha)}$$

If we compare the result of R_1 and R_2, me can clearly see the numerical difference between them.

$\alpha = \frac{1}{2}$	R_1= 0.500000000	$\alpha = \frac{1}{2}$	R_2= 0.276022378
$\alpha = \frac{1}{3}$	R_1= 0.369850263	$\alpha = \frac{1}{3}$	R_2= 0.274206244
$\alpha = \frac{1}{4}$	R_1= 0.353553390	$\alpha = \frac{1}{4}$	R_2= 0.278813286
$\alpha = \frac{1}{5}$	R_1= 0.329876977	$\alpha = \frac{1}{5}$	R_2= 0.271240978
$\alpha = \frac{1}{6}$	R_1= 0.314980262	$\alpha = \frac{1}{6}$	R_2= 0.270014934
$\alpha = \frac{1}{7}$	R_1= 0.304753413	$\alpha = \frac{1}{7}$	R_2= 0.268922646
$\alpha = \frac{1}{8}$	R_1= 0.297301778	$\alpha = \frac{1}{8}$	R_2= 0.267933365
$\alpha = \frac{1}{9}$	R_1= 0.291632259	$\alpha = \frac{1}{9}$	R_2= 0.267060422
$\alpha = \frac{1}{10}$	R_1= 0.287174588	$\alpha = \frac{1}{10}$	R_2= 0.266260272
$\alpha = \frac{1}{11}$	R_1= 0.28357813	$\alpha = \frac{1}{11}$	R_2= 0.265531749
$\alpha = \frac{1}{12}$	R_1= 0.280615512	$\alpha = \frac{1}{12}$	R_2= 0.264865773

References

[1] A.W., Goodman (1983), "*Univalent functions volume I and volume II.*" Tampa Florida Mariner publishing comp.

[2] C.H., Pommerenke (1975), "*Univalent functions with a chapter on quadratic differentials, by Gend Jansen*", Studia Mathematica Lehrbucher andenhoecek.

[3] R.J. Libera., (1967) "*Univalent - spiral functions*" Canad. J. Math (19) 449-456.

[4] M.S., Roberthson, (1968) , "*Univalent functions f(z) for which zf'(z) is spiralike*", Michigan Math. J. (14) 97-101.

[5] S.D., Bernardi, (1974), "*New distortion theorems for functions of positive real part and application*" (34) 113-118.

[6] P.L. Duren., (1983) "*Univalent functions*" Springer-Verlag.