The numerical solution of the singular two-point boundary value problems by using non-polynomial spline functions

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Abstract: The non-polynomial spline method is proposed to solve a singular boundary value problems. Some model problems are solved and the numerical results are compared with B-spline approximation and the exact solution. Results obtained by the method indicate the method is simple and effective.

Key–Words: Singular point; Singular two-point boundary value problems; Cubic spline

1 Introduction

We consider singular boundary value problems of the form [1,4]:

$$y^{''}(x) + \frac{k}{x} y^{'}(x) + b(x)y(x) = c(x) \quad (1)$$

with the following boundary conditions

$$y^{'}(0) = 0, y(1) = \beta \quad (2)$$

where $0 < x < 1$, and $b(x), c(x)$ are given continuous functions.

The existence and approximations of the solutions to non-linear systems of second-order BVPs have investigated by many authors [1-6]. In [1] parametric spline method is presented for a class of singular two point boundary value problems and they obtain classes of methods for different value of parameters. Jamet [2] considered a standard three-point finite-difference method in uniform mesh and has shown that the order is o($h^{1-\alpha}$), in maximum norm. In [3] a new three-point finite difference method based on uniform mesh is presented for a class of singular two-point boundary value problem. Caglar [4] modified the original differential equation at singular point then the boundary value problem was treated by using B-spline approximation. Reddien [5] and Reddin and Schumaker [6] used certain projection methods and singular splines to solve the linear problem and also studied the existence and uniqueness of solution.

The section of this paper are organized as follows: In the next section we describe the basic formulation of the spline function required for our subsequent development. In section 3 the method are used to analysis to solution of problem (1) and (2). In section 4 some numerical result, that are illustrated using MATLAB 6.5, are given to clarify the method. Section 5 ends this paper with a brief conclusion. Note that we have computed the numerical results by MATLAB 6.5.

2 Spline method

We divide the interval $[a, b]$ into n equal subintervals using the grid points

$$x_i = a + ih, i = 0, 1, 2, ..., n,$$

with

$$a = x_0, x_n = b, h = (b - a)/n$$

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where n is an arbitrary positive integer.

Let $u(x)$ be the exact solution and u_i be an approximation to $u(x_i)$ obtained by the non-polynomial cubic $S_i(x)$ passing through the points (x_i, u_i) and (x_{i+1}, u_{i+1}), we do not only require that $S_i(x)$ satisfies interpolatory conditions at x_i and x_{i+1}, but also the continuity of first derivative at the common nodes (x_i, u_i) are fulfilled.
We write $S_i(x)$ in the form:

$$S_i(x) = a_i + b_i(x - x_i) + c_i sin\tau(x - x_i) + d_i cos\tau(x - x_i), i = 0, 1, ..., n - 1 \quad (3)$$

where a_i, b_i, c_i and d_i are constants and τ is a free parameter.

A non-polynomial function $S(x)$ of class $C^2[a, b]$ interpolates $u(x)$ at the grid points $x_i, i = 0, 1, 2, ..., n$, depends on a parameter τ, and reduces to ordinary cubic spline $S(x)$ in $[a, b]$ as $\tau \to 0$. To derive expression for the coefficients of Eq. (3) in term of u_i, u_{i+1}, M_i and M_{i+1}, we first define:

$$S_i(x_i) = u_i, S_i(x_{i+1}) = u_{i+1},$$
$$S^{''}(x_i) = M_i, S^{''}(x_{i+1}) = M_{i+1}.$$

From algebraic manipulation, we get the following expression:

$$\begin{aligned} a_i &= u_i + \frac{Mi}{\tau^2}, \\ b_i &= \frac{u_{i+1} - u_i}{h} + \frac{M_{i+1} - M_i}{\tau\theta}, \\ c_i &= \frac{M_i cos\theta - M_{i+1}}{\tau^2 sin\theta}, \\ d_i &= -\frac{M_i}{\tau^2}, \end{aligned}$$

where θ=τh and $i = 0, 1, 2, ..., n - 1$.

Using the continuity of the first derivative at (x_i, u_i), that is $S'_{i-1}(x_i) = S'_i(x_i)$ we obtain the following relations for i=$1, ..., n - 1$.

$$\alpha M_{i+1} + 2\beta M_i + \alpha M_{i-1} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \quad (4)$$

where $\alpha = (-1/\theta^2 + 1/\theta \sin\theta)$, $\beta = (1/\theta^2 - \cos\theta/\theta\sin\theta)$ and $\theta = \tau h$. The method is fourth-order convergent if $1 - 2\alpha - 2\beta = 0$ and $\alpha = 1/12$ [7].

3 Analysis of the method

To illustrate the application of the Spline method developed in the previous section we consider the singular boundary value problems that is given in Eq. (1). At the grid point (x_i, u_i), the proposed singular boundary value problems in Eq. (1) may be discretized by

$$y_i^{''} + \frac{k}{x_i} y_i^{'} + b(x_i)y_i = c(x_i) \quad (5)$$

Substituting $M_i = y^{''}$ in equation (5):

$$M_i + \tfrac{k}{x_i} y_i^{'} + b(x_i)y_i = c(x_i)$$

$$M_i = -\frac{k}{x_i} y_i^{'} - b(x_i)y_i + c(x_i) \quad (6)$$

$$M_{i-1} = -\frac{k}{x_{i-1}} y_{i-1}^{'} - b(x_{i-1})y_{i-1} + c(x_{i-1}) \quad (7)$$

$$M_{i+1} = -\frac{k}{x_{i+1}} y_{i+1}^{'} - b(x_{i+1})y_{i+1} + c(x_{i+1}) \quad (8)$$

The following approximations for the first-order derivative of y in Eq. (6,7,8) can be used

$$y_{i+1}^{'} \cong \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h}, y_i^{'} \cong \frac{y_{i+1} - y_{i-1}}{2h}, \\ y_{i-1}^{'} \cong \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h},$$

So Eq. (6,7,8) becomes

$$M_i = -\frac{k}{x_i} \frac{y_{i+1} - y_{i-1}}{2h} - b(x_i)y_i + c(x_i), \quad (9)$$

$$M_{i+1} = -\frac{k}{x_{i+1}} \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h} - b(x_{i+1})y_{i+1} + c(x_{i+1}) \quad (10)$$

$$M_{i-1} = -\frac{k}{x_{i-1}} \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}$$

$$-b(x_{i-1})y_{i-1}+c(x_{i-1}) \qquad (11)$$

Substituting Eqs. (9-11) in Eqs. (4), we find the following $(n-1)$ linear algebraic equations in the $(n+1)$ unknowns for $i=1,2,3,...,n-1$.

$$\begin{aligned} & [\frac{-3\alpha k}{2hx_{i+1}}-\alpha b(x_{i+1})-\frac{2\beta k}{2hx_i}+\frac{\alpha k}{2hx_{i-1}}-\frac{1}{h^2}]y_{i+1} \\ + & [\frac{4\alpha k}{2hx_{i+1}}-2\beta b(x_i)-\frac{4\alpha k}{2hx_{i-1}}+\frac{2}{h^2}]y_i \\ + & [\frac{-\alpha k}{2hx_{i+1}}+\frac{3\alpha k}{2hx_{i-1}}+\frac{2\beta k}{2hx_i}-\alpha b(x_{i-1})-\frac{1}{h^2}]y_{i-1} \\ = & -\alpha c(x_{i-1})-2\beta c(x_i)-\alpha c(x_{i+1}) \end{aligned}$$

Since $x=0$ is singular point of Eq. (1),we modify Eq. (1) at $x=0$. By L'Hopital rule, the boundary value problem (1) is transform

$$y^{''}(x)=-\frac{b(x)y(x)}{k+1}+\frac{c(x)}{k+1}, \qquad (12)$$

Substituting $M_i=y^{''}$ in equation (12):

$$M_i=-\frac{b(x_i)y_i}{k+1}+\frac{c(x_i)}{k+1}$$

for $i=1$ in equation (4),

$$\begin{aligned} & y_2(\frac{-\alpha b(x_2)}{k+1}-\frac{1}{h^2})+y_1(\frac{-2\beta b(x_1)}{k+1}+\frac{2}{h^2}) \\ + & y_0(\frac{-\alpha b(x_0)}{k+1}-\frac{1}{h^2})=\frac{-\alpha c(x_2)}{k+1}-\frac{\beta c(x_1)}{k+1} \\ - & \frac{\alpha c(x_0)}{k+1} \end{aligned}$$

We need two more equations. The two end conditions can be derivated as follows:

$$y^{'}(0)=0, y(1)=0$$

4 Numerical examples

In this section, to illustrate our methods we have solved two singular boundary value problems. All computations are done by using MATLAB 6.5.

Example 1.

Consider the Bessel's equation of order zero

$$y^{''}(x)+\frac{1}{x}y^{'}(x)+y(x)=0$$

subject to the boundary conditions

$$y^{'}(0)=0, y(1)=1.$$

The analitical solution is $y(x)=\frac{J_0(x)}{J_0(1)}$. The numerical results are given in Table 1 and Table 2.

Example 2.

We consider the following equations

$$y^{''}(x)+\frac{2}{x}y^{'}(x)-4y(x)=-2$$

subject to the boundary conditions

$$y^{'}(0)=0, y(1)=5.5$$

where $0<x\le 1$. The exact solutions of $y(x)$ is given as $0.5+\frac{5sinh2x}{xsinh2}$.The numerical results are given in Table 3 and Table 4.

Table 1: y_i :Spline solution, y_{1i}:B-spline solution,Y_i analitical solution .

x_i	$y_{i(1/20)}$	$y_{1i(1/20)}$	Y_i
0.000	1.302346	1.306823	1.306956
0.050	1.302346	1.306007	1.306139
0.100	1.300718	1.303558	1.303691
0.300	1.276028	1.277587	1.277714
0.500	1.225495	1.226420	1.226539
0.700	1.151102	1.151585	1.151690
0.900	1.055166	1.055309	1.055394
1	1.000000	1.000000	1.000071

Table 2: The maximum absolute error for $y(x)$ for different value of nodal point from example 1.

h	y
1/10	0.0148
1/20	0.0045
1/40	0.0013
1/60	6.3842e-004
1/120	1.8117e-004
1/520	1.2073e-005

Table 3: y_i :Spline solution, y_{1i}:B-spline solution,Y_i analitical solution .

x_i	$y_{i(1/40)}$	$y_{1i(1/40)}$	Y_i
0.000	3.2604	3.257131	3.257205
0.050	3.2627	3.261729	3.261803
0.100	3.2761	3.275550	3.275624
0.300	3.4258	3.425570	3.425642
0.500	3.7404	3.740206	3.740271
0.700	4.2505	4.250342	4.250393
0.900	5.0068	5.006744	5.006767
1	5.5000	5.499999	5.500000

Table 4: The maximum absolute error for $y(x)$ for different value of nodal point from example 2. .

h	y
1/10	0.0290
1/20	0.0078
1/40	0.0020
1/60	9.1046e-004
1/120	2.3059e-004
1/520	1.2404e-005

5 Conclusion

In this paper, the non-polynomial spline method is developed for the approximate solution of nonlinear system of the second-order boundary value problems.table 1 and table 3, we have tabulated the exact solution together with solutions given by B-spline and non-polynomial cubic spline methods are as exhibited before. In table 2 and table 4, we have calculated absolute error of non-polynomial cubic spline method compare with the exact solution.The results of the test examples show that the non-polynomial spline method results are equal to B-spline results. The numerical results obtained by using the method described in this study give acceptable results. We have concluded that numerical results converge to the exact solution when h goes to zero. Use of spline method has show that it is an applicable method for singular boundary value problems.

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