

EXPERIMENTAL STUDY OF A NONLINEAR CIRCUIT DESCRIBED BY DUFFING'S EQUATION

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Abstract

We have studied experimentally an electronic circuit that implements the Duffing equation. The circuit appears periodic and non-periodic (chaotic) dynamics behavior, as we vary the amplitude of the driving voltage signal V_0 . The expected operation of the circuit was confirmed, by comparing the experimental results with the results of the simulation. From the study of the circuit's behavior, very important phenomena concerning the Chaos theory were detected, such as the great sensitivity of the circuit to initial conditions, the route to chaos through the mechanism of period doubling and the phenomenon of crisis of chaotic attractors.

Keywords: *Chaos, Duffing equation, Phase portrait, Poincare map, Bifurcation diagram, Period doubling, Crisis of attractor.*

1. Introduction

Most people use the term "chaos", in order to attribute to a phenomenon, accidental behavior. Thought scientifically, this is not right. The strict scientific definition of chaos includes deterministic elements, making these two senses (chaos and determinism) correlative. As much as it seems strange, chaos has rules that give to it, structure and order. Chaos scientifically is specified as the great sensitivity of a non-linear system to initial conditions. That is, the smallest variation of the initial conditions of a chaotic system can bring great variation of its future condition.

The science of Chaos is relatively new, though Chaos had already been observed from Van der Pol in 1927 [1]. Since then, many scientists have observed the chaotic behavior in many physical systems. However very often, scientists considered this behavior undesirable, something that they could not explain and often they thought, it was because of noise or experimental mistakes. This involuntary lapse was expected, since linear equations, that are used to model a system, could not predict Chaos. Chaotic behavior is predicted, only when non-linearity, which is the source of this behavior, is included in the model.

Since 1963, when Lorenz published his paper, concerning the prediction of weather [2], many chaotic systems have been described in many areas [3], such as electric circuits [4], chemistry and biochemistry [5], economy [6] etc. Probably the best way to introduce Chaos is through the tools that we use to study it. Two very important tools in order to observe chaos are Phase portraits and Poincare maps.

In this paper, we will present a circuit that implements one of the well known differential second order equations, Duffing equation. This equation exhibits a variety of phenomena, which are related with Chaos theory, such as the dependence of a system on initial conditions, the crisis of chaotic attractors and the route to chaos through period doubling. These phenomena were experimentally confirmed from the operation of the system, while the right operation of the circuit was confirmed by comparing the experimental data with the results of the simulation.

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2. The Duffing-type Circuit

Duffing's equation,

$$\frac{d^2 x_1}{dt^2} + \varepsilon \cdot \frac{dx_1}{dt} + a \cdot x_1 + b \cdot x_1^3 = u(t), \quad (1)$$

is one of the most famous and well studied nonlinear non-autonomous equations, exhibiting various dynamic behaviors, including Chaos and bifurcations. One of the simplest implementations of the Duffing equation has been presented by Leuciuc [8]. It is a second order nonlinear circuit, which is excited by a sinusoidal voltage source ($u(t)=V_0 \cos(\omega t)$), and contains two op-amps (LF411) operating in the linear region (Figure 1).

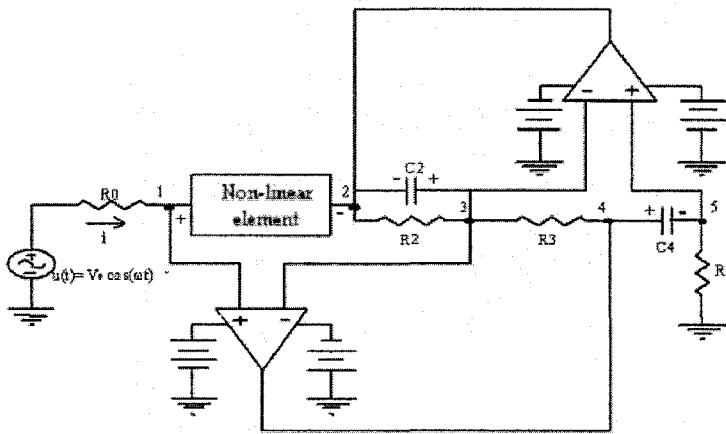


Figure 1. The electronic circuit obeying Duffing's equation.

This circuit has also a very simple nonlinear element (Figure 2(b)), implementing a cubic function of the form

$$i(v) = p \cdot v + q \cdot v^3 \quad (2)$$

which is shown in Figure 2(a).

The values of the resistors R_{11} , R_{12} , specify the grade M_1 , while the resistor R_1 specifies the grade M_0 . If we use the same resistors, $R_{11}=R_{12}$, the characteristic curve has symmetry, as you can see in Figure 2(a). The diodes we used are LF411.

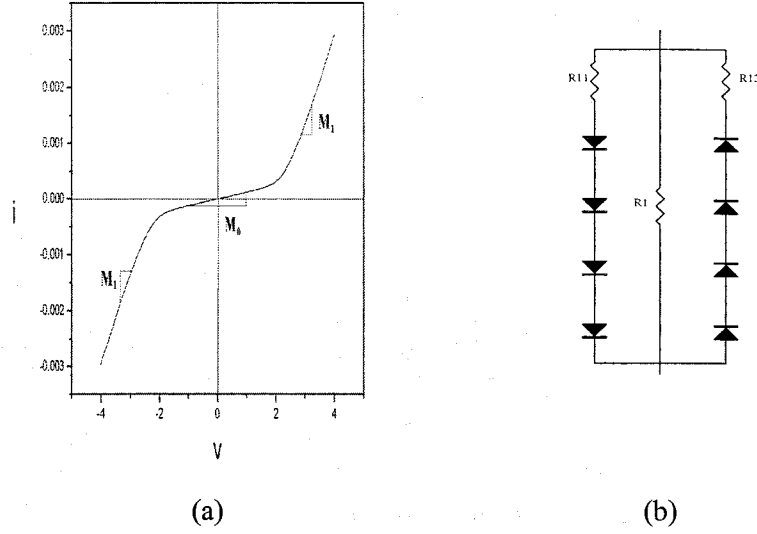


Figure 2. The nonlinear element implementing the cubic function of the form

$$i(v) = p \cdot v + q \cdot v^3.$$

Denoting by x_1 and x_2 the voltages across capacitors C_2 and C_4 respectively, we have the following state equations.

$$\frac{dx_1}{dt} = -\frac{1}{C_2 \cdot R_2} \cdot x_1 + \frac{1}{C_2 \cdot R_3} \cdot x_2 \quad (3)$$

$$\frac{dx_2}{dt} = -\frac{R_0}{C_4 \cdot R_5} \cdot f(x_1) + \frac{V_0}{C_4 \cdot R_5} \cdot \cos(\omega \cdot t) \quad (4)$$

where, $f(x_1) = p \cdot x_1 + q \cdot x_1^3$, is a cubic function.

Finally, from equations (3) and (4), we take the Duffing equation (1), where,

$$\epsilon = \frac{1}{C_2 \cdot R_2}, \quad a = \frac{p \cdot R_0}{C_2 \cdot C_4 \cdot R_3 \cdot R_5}$$

$$b = \frac{r \cdot R_0}{C_2 \cdot C_4 \cdot R_3 \cdot R_5}, \quad B = \frac{V_0}{C_2 \cdot C_4 \cdot R_3 \cdot R_5}$$

The values of circuit parameters are $R_0=2.05\text{k}\Omega$, $R_2=5.248\text{k}\Omega$, $R_3=R_5=1\text{k}\Omega$, $R_{11}=R_{12}=0.557\text{k}\Omega$, $R_1=8.11\text{k}\Omega$, $C_2=105.9\text{nF}$, $C_4=9.79\text{nF}$, $f=1.273\text{kHz}$, while the amplitude V_0 ranges from [1.6, 3.2] (Volt), so the normalized parameters take the following values $a=0.25$, $b=1$, $\epsilon=0.18$, $\omega=0.8$ and B ranges from [16, 32].

3. Dynamic behavior of the circuit

A very important tool for the study of the dynamic behavior of the circuit was the program we wrote in Pascal using the Runge-Kutta algorithm. This program solves arithmetically the Duffing equation and we used it to take the theoretical Phase portraits and the Poincare maps. Also, with this program we took the Bifurcation diagram (Figure 3).

The Bifurcation diagram is very useful because it help us to mark the regions where the system appears periodic and non-periodic (chaotic) behavior. So, from the Bifurcation diagram of Figure 3, arises a Table 1, where we can see the dynamic behavior of the system at the range of $V_0 \in [1.6 - 3.2]$ (Volt).

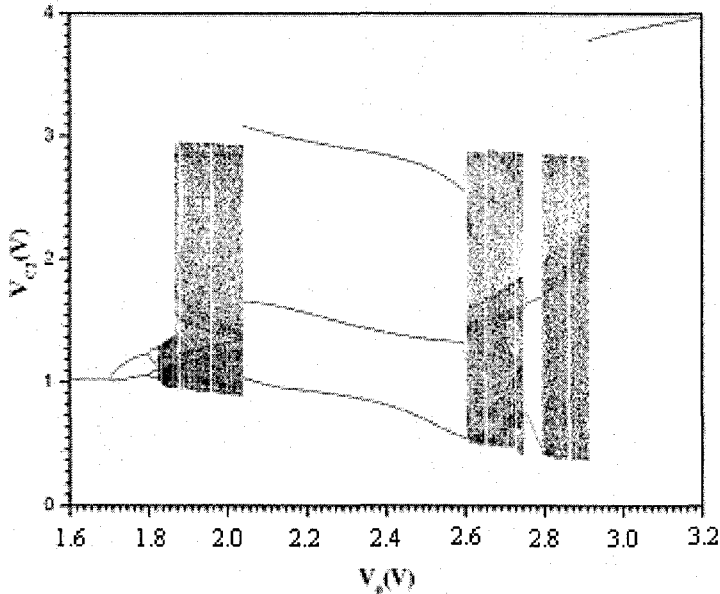


Figure 3. Bifurcation diagram from $V_0 \in [1.6 - 3.2]$ (Volt).

Table 1. Dynamic behavior of the system for $V_0 \in [1.6 - 3.2]$ (Volt).

$V_0 [,)$ (Volt)	Behavior	$V_0 [,)$ (Volt)	Behavior
1.600-1.704	period-1	2.652-2.656	period-4
1.704-1.802	period-2	2.656-2.658	period-8
1.802-1.820	period-4	2.658-2.728	chaos
1.820-1.824	period-8	2.728-2.732	period-4
1.824-1.878	chaos	2.732-2.748	chaos
1.878-1.882	period-5	2.748-2.756	period-4
1.882-1.958	chaos	2.756-2.794	period-2
1.958-1.964	περίοδο-6	2.794-2.798	period-4
1.964-2.040	chaos	2.798-2.860	chaos
2.040-2.606	period-3	2.860-2.870	period-5
2.606-2.650	chaos	2.870-2.914	chaos
2.650-2.652	period-2	2.914-3.200	period-1

As we can see in Table 1, the system for values of amplitude $V_0 \in [1.6 - 1.704]$, has a periodic behavior. Specifically, the system is in period-1 steady state. This means that a

period T_r , of the response signal is equal to the period T_s , of the signal of the external sinusoidal source. This behavior in the Bifurcation diagram is represented with a single line. From the simulation with the program, we took the Poincare map (circle), and a Phase portrait (line), (Figure 4(a)), while we compared it with an experimental Phase portrait, which we took from the oscilloscope (Figure 4(b)). When the system has a periodic behavior, we know that the Phase Portrait is a closed curve and the Poincare map has a specific number of points (e.g. period-1 \rightarrow 1 point, period-2 \rightarrow 2 points etc).

From the Table 1 we saw, that the circuit is in a period-2 steady state, for $V_0 \in [1.704 - 1.802)$. That is a period T_r , of the response signal is twice a period T_s , of the signal of the external sinusoidal source ($T_r = 2T_s$). So, we choose a value for the amplitude in the above range, $V_0 = 1.75$ Volt, and we took again the Phase Portrait and the Poincare map, from the simulation, as we can see in Figure 5(a). The Poincare map produces two points as we expected. The experimental Phase portrait (Figure 5(b)) confirms the exact operation of the circuit.

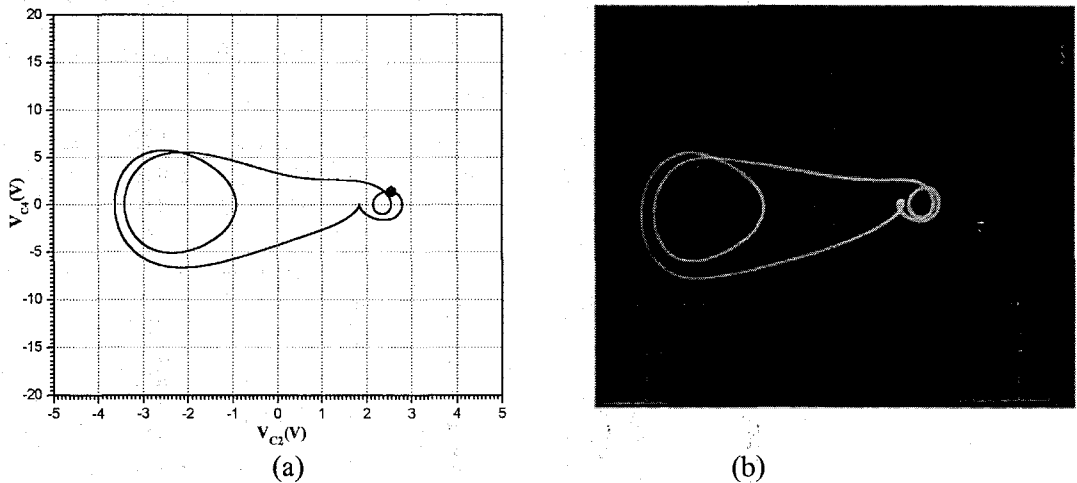


Figure 4. (a) Phase portrait (closed curve) and Poincare map (point) for $V_0 = 1.64$ Volt (period-1), (b) Experimental Phase portrait for $V_0 = 1.64$ Volt (period-1) (Horiz. V_{C2} : 1V/div., Vert. V_{C4} : 5V/div.).

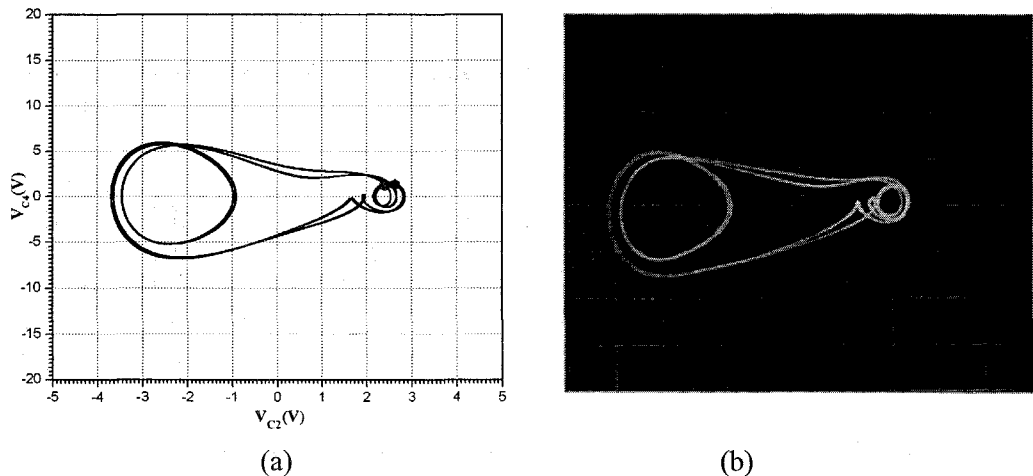


Figure 5. (a) Phase portrait (closed curve) and Poincare map (points) for $V_0 = 1.75$ Volt (period-2), (b) Experimental Phase portrait for $V_0 = 1.75$ Volt (period-2) (Horiz. V_{C2} : 1V/div., Vert. V_{C4} : 5V/div.).

For $V_0 = 1.81$ Volt, the circuit is in a period-4 steady state. So, in Figure 6(a), we turn up four points in the Poincare map, and the circuit has the desirable behavior again, as we can see for the comparison of the two Phase portraits (simulation and experimental).

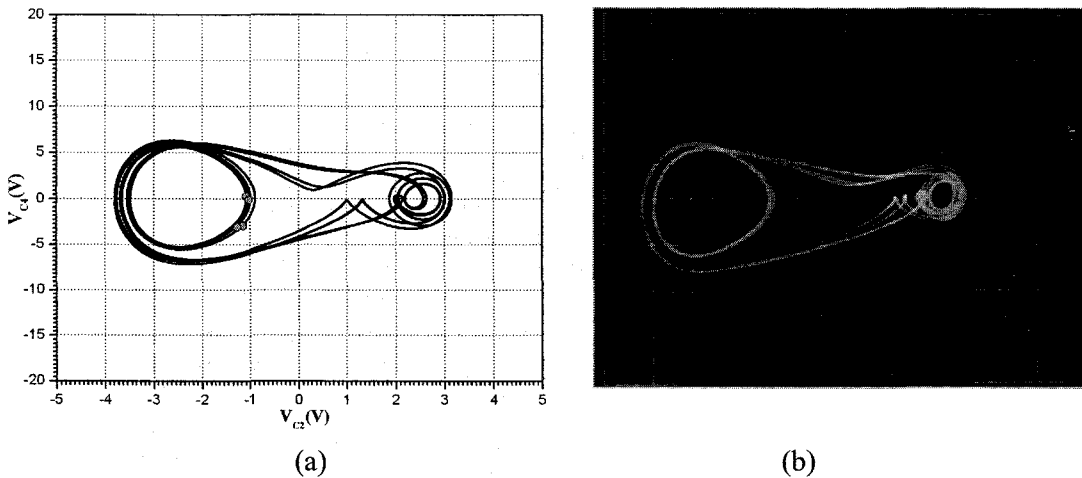


Figure 6. (a) Phase portrait (closed curve) and Poincare map (points) for $V_0=1.81$ Volt (period-4), (b) Experimental Phase portrait for $V_0=1.81$ Volt (period-4) (Horiz. V_{C2} : 1V/div., Vert. V_{C4} : 5V/div.).

So we observe that as the value of the amplitude increases, the «period» of the circuit doubles. Therefore, the circuit appears the phenomenon of «period doubling» and finally inserts to chaos for $V_0 = 1.824$ Volt. The first who studied this phenomenon was R. M. May [9].

Also, from the Bifurcation diagram (Figure 3) we can see an enlargement of the chaotic region at the value of $V_0 = 1.868$ Volt. This phenomenon is very common and called «interior crisis of the chaotic attractor» [10]. In Figure 7 we exhibit two experimental phase portraits where you can see the enlargement of the chaotic region.

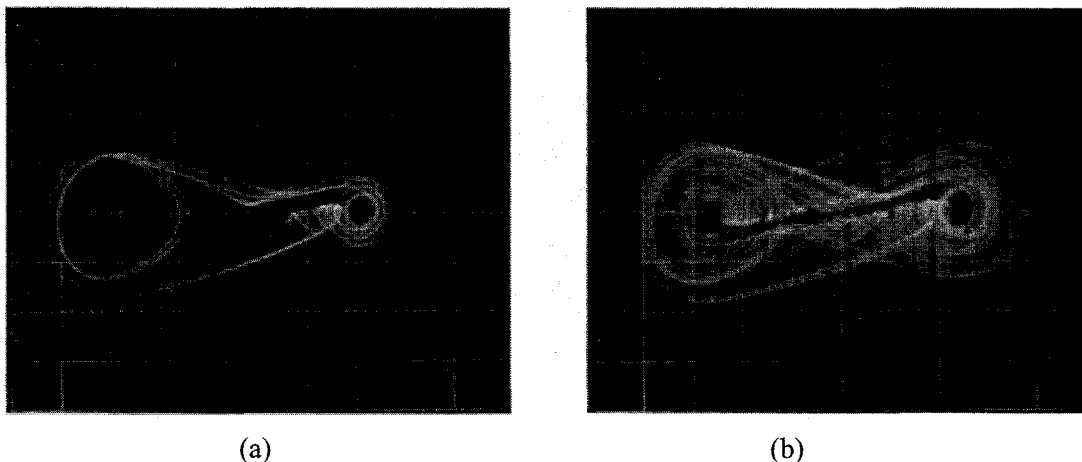


Figure 7. (a) Experimental Phase portrait for $V_0=1.85$ Volt (chaotic behavior) (b) Experimental Phase portrait for $V_0=1.95$ Volt. (expansion of the chaotic region) (Horiz. V_{C2} : 1V/div., Vert. V_{C4} : 5V/div.).

For $V_0 \in [1.824 - 2.04]$ Volt, the circuit has a chaotic behavior, which it is intercepted by small periodic windows. In Figure 8 we can see the comparison of the two Phase portraits (simulation – experimental), while in Figure 9 we can see the chaotic attractor for $V_0 = 2$ Volt.

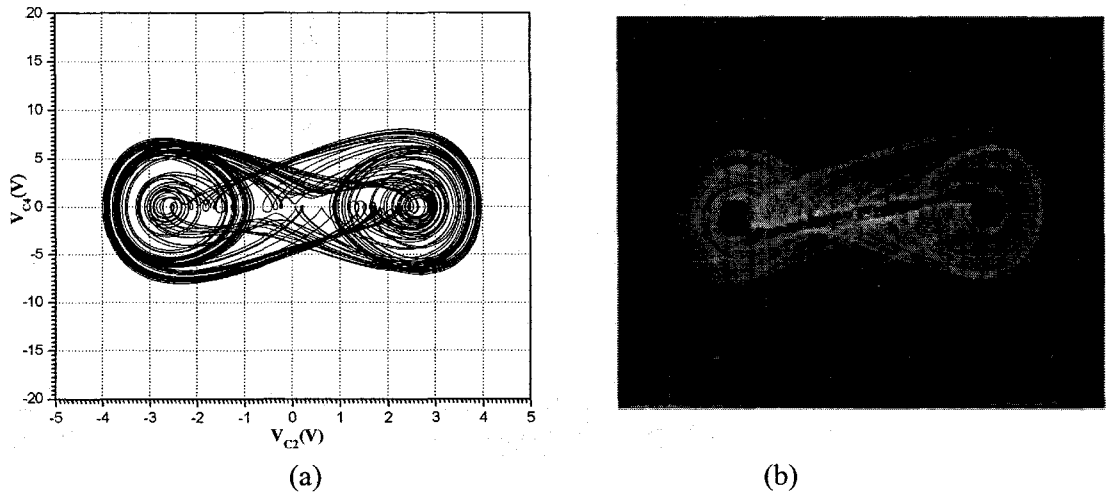


Figure 8. (a) Phase portrait for $V_0=2$ Volt (Chaotic behavior), (b) Experimental Phase portrait for $V_0=2$ Volt (Chaotic behavior) (Horiz. V_{C2} : 1V/div., Vert. V_{C4} : 5V/div.).

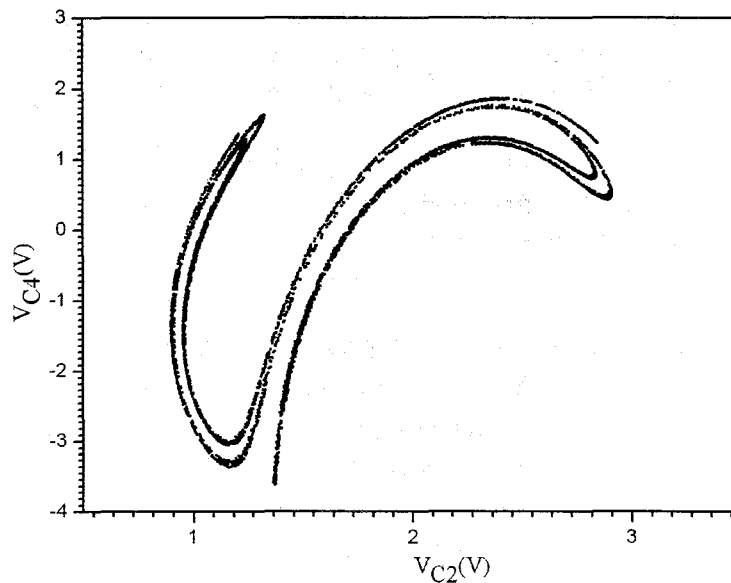


Figure 9. The chaotic attractor for $V_0=2$ Volt.

While the value of the amplitude V_0 increases, the circuit goes out from the chaos and comes in the region of periodic behavior (period-3) for $V_0 \in [2.04 - 2.606]$ Volt, (Figure 10).

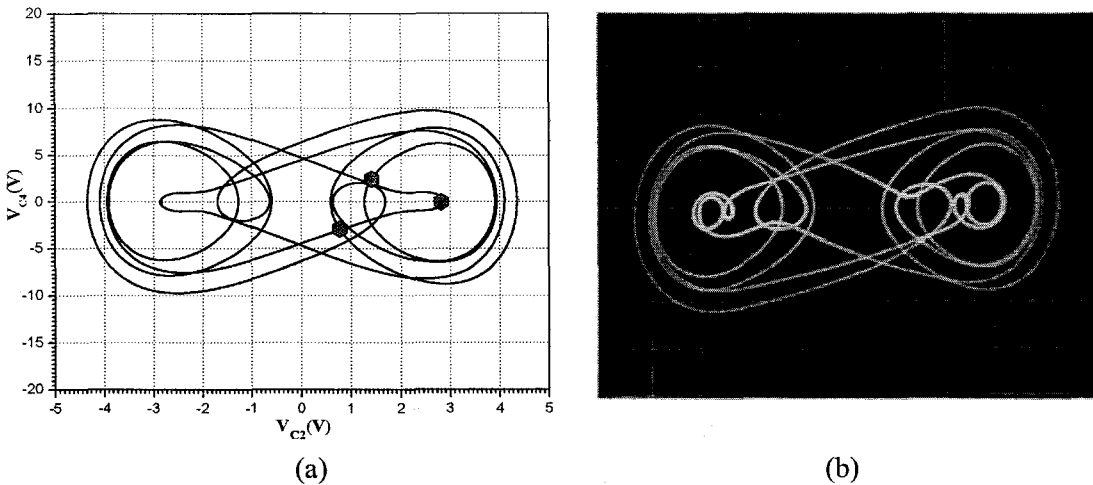


Figure 10. (a) Phase portrait (closed curve) and Poincare map (points) for $V_0=2.4$ Volt (period-3), (b) Experimental Phase portrait for $V_0=2.4$ Volt (period-3)

(Horiz. V_{C2} : 1V/div., Vert. V_{C4} : 5V/div.)

For V_0 greater than 2.606 Volt, the circuit comes in another chaotic region, with small windows of periodic behavior. In Figure 11 we can see the phase portraits for one of these windows (period-2), for $V_0 \in [2.756 - 2.794]$ Volt.

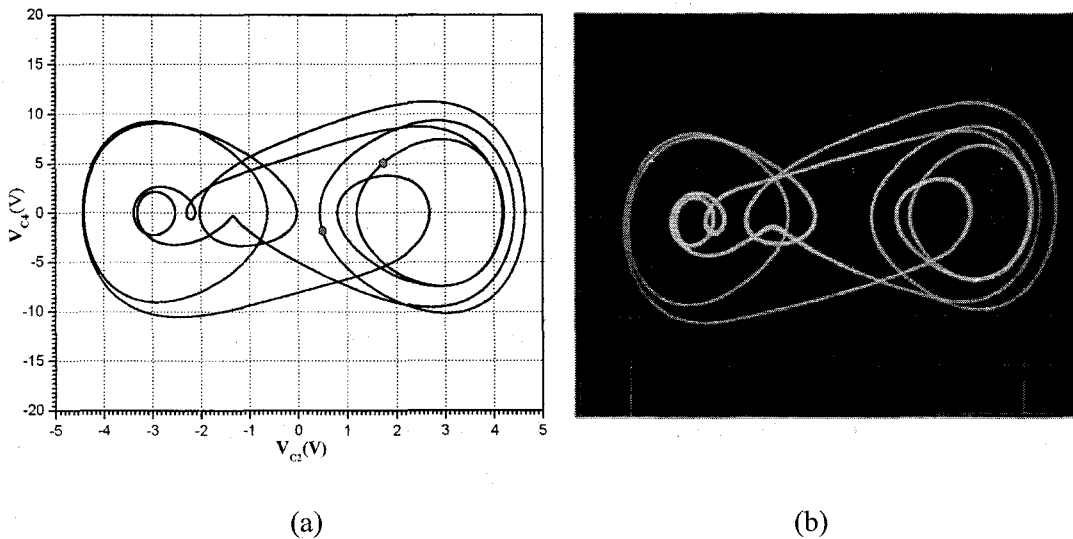


Figure 11. (a) Phase portrait (closed curve) and Poincare map (points) for $V_0=2.77$ Volt (period-2), (b) Experimental Phase portrait for $V_0=2.77$ Volt (period-2)
(Horiz. V_{C2} : 1V/div., Vert. V_{C4} : 5V/div.)

Finally, the circuit comes out from the chaotic region for $V_0 = 2.914$ Volt, and for greater values, shows periodic behavior (period-1), as we can observe in Figure 12.

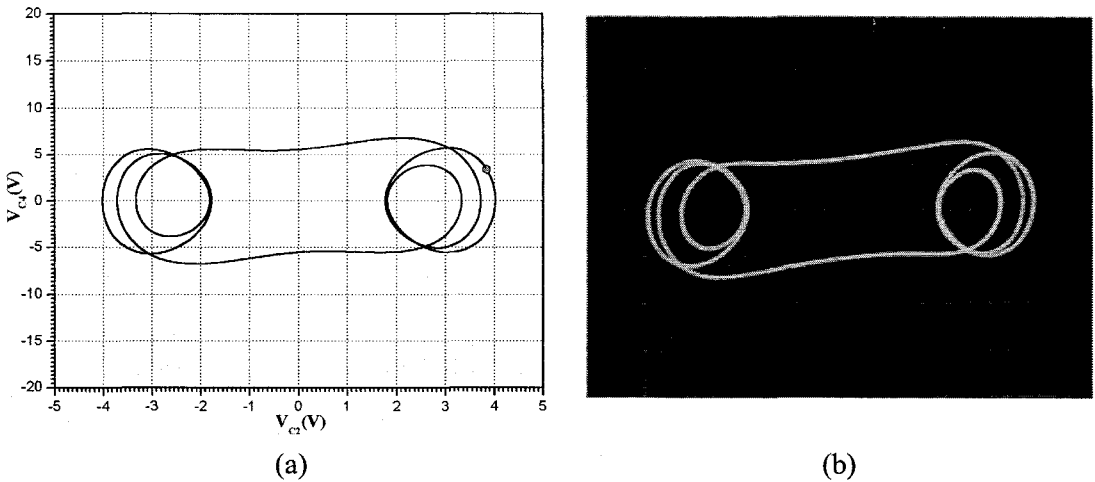


Figure 12. (a) Phase portrait (closed curve) and Poincaré map (point) for $V_0=3$ Volt (period-1), (b) Experimental Phase portrait for $V_0=3$ Volt (period-1) (Horiz. V_{C2} : 1V/div., Vert. V_{C4} : 5V/div.).

One characteristic example of the complexity of the circuit and its great sensitivity at the initial conditions is the fact that, for the same value of the parameters but with different initial conditions the circuit has entirely different dynamic behavior.

This fact is confirmed since the circuit for $V_0 = 2.8$ Volt has a chaotic behavior (Figure 13(a)), while the circuit for the same value of the amplitude but with different initial conditions has a periodic behavior (Figure 13(b)).

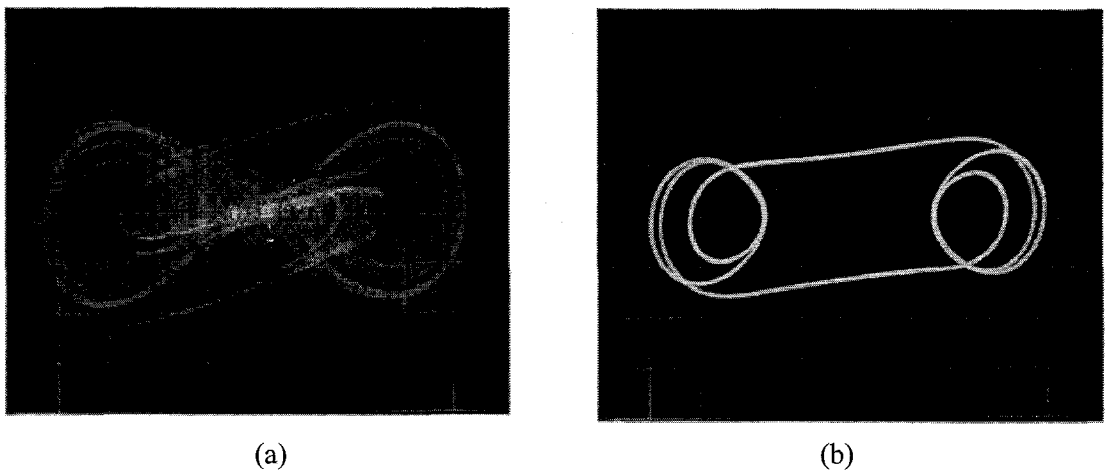


Figure 13. (a) Experimental Phase portrait for $V_0=2.8$ Volt. (chaotic behavior) (b) Experimental Phase portrait for $V_0=2.8$ Volt. (periodic behavior) (Horiz. V_{C2} : 1V/div., Vert. V_{C4} : 5V/div.).

This behavior is confirmed from the simulation process. As we can see at the Bifurcation diagram (zoom in the region $V_0 \in [2.7 - 3.2]$ Volt), in Figure 14(a), the circuit for $V_0 = 2.8$ Volt, has a chaotic behavior, but in Figure 14(b), where we can see the reverse Bifurcation diagram, the circuit for $V_0 = 2.8$ Volt, has a periodic behavior.

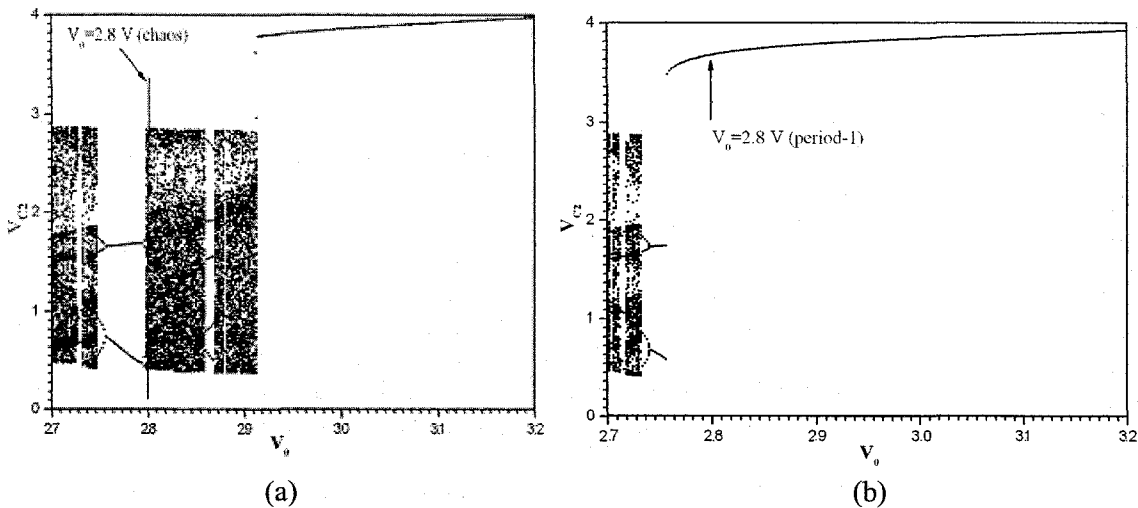


Figure 14. (a) Bifurcation diagram for $V_0 \in [2.7 - 3.2]$ Volt, ($V_0 = 2.8$ Volt \rightarrow chaos)
 (b) Reverse Bifurcation diagram for $V_0 \in [2.7 - 3.2]$ Volt ($V_0 = 2.8$ Volt \rightarrow period-1)

4. Conclusions

In this paper we have presented a circuit that implements the well known second order Duffing equation. We have studied this circuit theoretically, by simulation, and experimentally. From the comparison of these two approaches we conclude that the circuit has the desirable operation. Also we discovered very important phenomena concerning the Chaos theory, such as, the great sensitivity of the circuit at initial conditions, the route to chaos through the mechanism of period doubling, and the phenomenon of crisis of chaotic attractors.

Acknowledgements

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