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DOES A CHAOTIC SYSTEM DYNAMIC REALLY EXIST IN NATURE OR IS IT A MISCONCEPTION DYNAMICS? : A HYPOTHESIS

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Abstract

The theory of nonlinear dynamical systems deals with deterministic systems that exhibit a complicated and random-looking behavior. Life sciences have been one of the most applicable areas for the ideas of chaos because of the complexity of biological systems. It is widely appreciated that chaotic behavior dominates physiological systems. However, as an extension of this trend, a new hypothesis is proposed that the existence of embedded nonlinear systems suggest a new rationale fundamentally which is different from the classic approach. A biological system can be considered as a simple explanation of transitions breaking up generic orbits onto higher dimensions with covering maps by preventing chaos. We seek to discuss and understand how a biological system can decrease its vulnerability to sensitivity at system transitions what we define those transitions as injective immersions of differentiable smooth manifolds with each corresponding to a transition to different state like synchronization, anti-synchronization and oscillator-death when network structure varies abruptly and asynchronously. We can then consider a biological system if an existence of such a unique immersed smooth submanifold into higher dimensional space can be shown that there is no chaotic dynamics associated with a map from one manifold to another one when the system is perturbed. We then will introduce an open problem whether Melnikov function is a continuously decreasing function for small perturbations which this distance function serves as a discriminate function for implications of the chaos transitions.

1.Introduction

Biological systems are networks of coupled nonlinear oscillators manifesting controlled autonomous dynamics events like fast synchronization, locomotion, and training with the abrupt varying network topology with folding asynchronously in terms of intersecting stable and unstable manifolds with excitation of different periods. It is believed that that chaotic process is confined to a manifold of lower dimension even though in fact, it may cover very high space. We can find that there are varying tendencies along different directions of trajectories when studying several ones which are starting close to a given point with a given dimension of space. One of these trajectories can acquire a dominating behavior, while weaker ones can go along other directions with contractions around an attractor point. But a covering map can imbed this system onto higher dimension by compensating that contractions can divert from each other by making weaker trajectories in higher space under perturbation.

The individual biological systems are organized in hierarchy in different dimensions characterized by self-similar dynamics with long-range order operating over multiple spatiotemporal scales. We usually assume that the dissipative structure on a lower level of the hierarchy is in a quasi-stationary level when we go up to higher levels it becomes stationary. Thus some of the parameters vary slowly and we observe different stationary states which subsequently appear which are embedded into each other toward to higher space domains.

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The concept of chaos which applies to a biological process is described by a deterministic and mathematical framework. This study will not discuss with artificial chaotic sets whose maps are restricted to some small neighborhood of bifurcation point in those sets. Whereas, in a biologically system when a system is perturbed, the immersion becomes globally injective and every point of system is effected because the system comprises of coupled nonlinear oscillators running in different temporal scales distributed over extended phase dimensions.

A biological system highly depends on both different time and spatial scales. The examples of a biological system are genetics and genomics, polymorphisms and molecular aspects of evolution, signal transduction pathways and networks, stress responses, pharmacogenomics in cancer biology. The long term behavior of a chaotic process is very difficult to distinguish from a noisy process with regard to probabilistic aspects. There is no sharp border between chaos and noise. In other words, we observe a self-embedding continuity of smooth transitions of intersecting varying dimensional stable and unstable manifolds when we perturb the biological system with the increasing the complexity of biological system. If a biological system has large number of incoherent uncontrolled influences we say that we observe chaos. But we can go one step beyond chaos with complexity theory based on differential geometry and we can explain complex behavior of biological systems that emerges within dynamic nonlinear systems.

The dynamics of a biological system is characterized by trajectories of transitions between stable and unstable manifolds which give us information about behavior of the system. Those trajectories can be mixtures of periodic, quasiperiodic (harmonicities in populations) and chaotic trajectories. The problem is to discuss whether a singular bifurcation point which leads us a chaotic trajectory on a defined manifold can be removed by immersing this trajectory into higher dimensions of a topological space and thereby leading dissemination of chaotic trajectory into periodic and quasiperiodic trajectories with a continuous curve connecting stable and unstable points in a covering space at higher dimensions. A new manifold now again becomes path-connected. We define this event first in this paper as we will afterward call it "from less harmonicities (periodicity)-to-mapping onto higher extended phase coverings'. In other words, the more dynamics of biological system acquires higher quasiperiodicity, then the more system becomes distributed in spatial space where both topology and mapping functions go toward into more smoothness in higher dimensions thereby building a higher covering space with less chaos by imbedding (suspending) in the extended autonomous phase manifold.

2.Background on General Nonlinear Dynamical Representation of Biological Systems

We can consider a general biological system of first order coupled autonomous ordinary differential equations under defined a coupled network. We can represent an autonomous biological system without perturbation

$$\dot{x} = g(x) \qquad (2.1)$$

where state vector $\vec{x} = x(t) = (x_1, x_2, \cdots x_n)^T \in \Re^n$ is a vector function of the independent variable time living on manifolds (i.e. A, B). In a biological system, there is no explicit time dependence i.e. system is autonomous $g(x,t) \equiv g(x)$. $g = (g_1, g_2, \cdots g_n)^T$ is a smooth vector field. We assume that $g(x)$ is continuously differentiable. The system structure is defined on a some subset $U \subseteq \Re^n$. Overdot indicates first order differentiation with respect to time t. The vector field generates a flow $\phi_t : U \rightarrow \Re^n$ where $\phi_t(x) = \phi(x,t)$ is a smooth solution function of (2.1) defined for that every point **x** is contained in some open set U and t is an interval $I \subseteq \Re$. Initial conditions $x(t_0) = x_0 \in U$ can be written as $x(x_0, t_0; t)$. Then any biological system can be considered as the set of all solutions with initial conditions in the set $U \in \Re^n$. Then the function g is defined on this set and denotes time dynamics of the system.

For example, we can write the autonomous system (2.1) for two dimensional planar system as the solution of the nonlinear system as follows if we excite the system quasiperiodically. Then we can hypothesis that when a biological system is excited, the solution manifold of dimension $n = 2$ is suspended (covered) in the extended (autonomous) phase manifold $\{x_\alpha^1, x_\alpha^2, \theta_\alpha^1, \theta_\alpha^2, \cdots \theta_\alpha^k, \cdots \theta_\alpha^l\}$ in a covering space where $\theta_\alpha^i = \lfloor \omega_\alpha^i t + \theta_\alpha^{0i} \rfloor (\mathrm{mod}\, 2\pi)$. When a biological system goes into unstable manifold, it can be suspended (covered) it in the extended phase manifold by imbedding onto itself to reduce chaotic behaviour.

$$\dot{x}_\alpha = g(x_\alpha) + \sum_{\substack{l=1,2 \\ i=1,2,\ldots L}}^{L} d_f(x_\alpha^l, \theta_\alpha^i) \qquad (2.2)$$

$$\dot{\theta} = \omega$$

$\theta = [\theta_\alpha^1, \theta_\alpha^2, \cdots \theta_\alpha^l], \omega = [\omega_\alpha^1, \omega_\alpha^2, \cdots \omega_\alpha^l]^T$ are the coordinates of the extended phase system where l is defined in (2.4). Immersion operator d_f denotes exterior derivative of one degree over extended phase coordinates of geometrical coordinates.

3.Background in terms of differentiable manifolds and covering maps

Definition 3.1 (Differential manifold)

A set M is called a C^∞ differentiable manifold of dimension n if M is covered by domains (i.e. energy landscapes in biomolecular processes) of some family of coordinate mappings or charts $\{x_\alpha : U_\alpha \rightarrow \Re^n\}_{\alpha \in A}$ where $x_\alpha = \{x_\alpha^1, x_\alpha^2, \cdots x_\alpha^n\}$. We require that the coordinate change maps $x_\beta \circ x_\alpha^{-1}$ are continuously differentiable any number of times on their natural domains in $\Re^n$. We require that the functions [3].

$$\begin{aligned} x_\beta^1 &= x_\beta^1(x_\alpha^1, \cdots, x_\alpha^n) \\ x_\beta^2 &= x_\beta^2(x_\alpha^1, \cdots, x_\alpha^n) \qquad (2.3) \\ &\vdots \\ x_\beta^n &= x_\beta^n(x_\alpha^1, \cdots, x_\alpha^n) \end{aligned}$$

together give a C^∞ bijection. α and β are indices selected from some set A for naming the individual charts. When we evaluate a functional relation at a point p on the manifold

$$(x_\beta^1(p),\cdots x_\beta^n(p)) = x_\beta \circ x_\alpha^{-1}(x_\alpha^1(p),\cdots,x_\alpha^n(p))$$

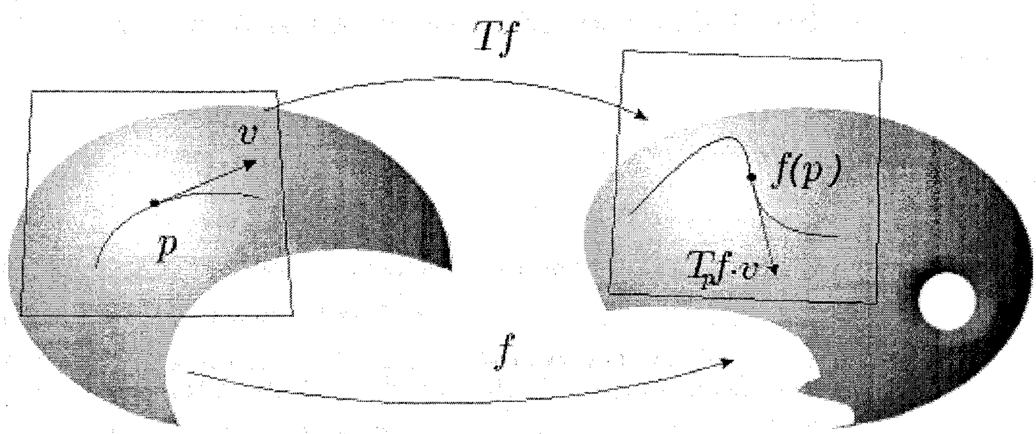
Definition 3.2(coordinate system)

Let M be a topological manifold. A pair (U,x) where U is an open subset of M and $x: U \to \Re^n$ is a homeomorphism is called a coordinate system (chart) on M.

For example any manifold M we can construct the "cylinder" $M \times I$ where I is some interval in $\Re$.

Definition 3.3 (tangent map)

Tangent map of a map is a linear map. We denote as $f: M, p \to N, f(p)$. We give a f and $p \in M$ and define a tangent map at bifurcation point p. $T_p f: T_p M \to T_{f(p)} N$.



If we have a smooth mapping function between manifolds $f: M \to N$ and we consider a point $p \in M$ and its image $q = f(p)$ then we define the tangent map at p by choosing any chart $(x;U)$ containing p and a chart $(y;V)$ containing $q = f(p)$ and then for any $v \in T_p M$ we have the representative $dx(v)$ with respect to $(x;U)$: Then the representative of $T_p(v).f$ is given by $dy(T_p(v).f) = D(y \circ f \circ x^{-1})dx(v)$. This uniquely determines $T_p(f).v$ and the chain rule guarantees that this is well defined (independent of the choice of charts) [3].

Definition 3.4 (Atlas)

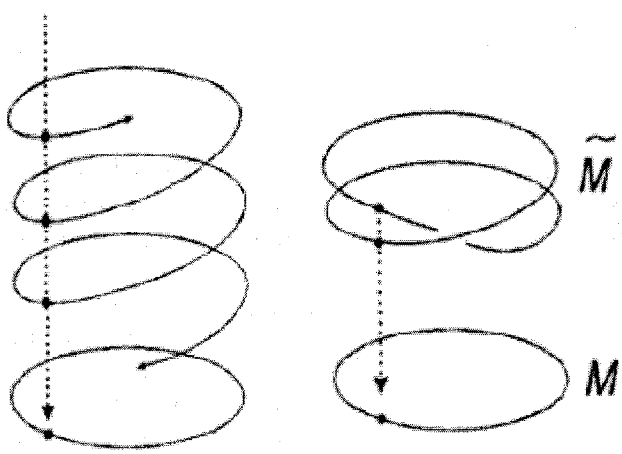
Let $A = \{(x_\alpha, U_\alpha)\}_{\alpha \in A}$ be an atlas Γ on a topological manifold M. Whenever the overlap $U_\alpha \bigcap U_\beta$ between two coordinate systems is nonempty we have the change of coordinates map $x_\beta \circ x_\alpha^{-1}: x_\alpha(U_\alpha \bigcap U_\beta) \to x_\beta(U_\alpha \bigcap U_\beta)$. If all such change of coordinates maps are C^r –diffeomorphism then we call the atlas a C^r-atlas.

Definition 3.5 ("transition maps" or "coordinate map changes")

Let Γ be a C^r a pseudogroup of transformations on a model space M. An atlas, for a topological space M is a family of charts(coordinate systems) $A_\Gamma = \{(x_\alpha, U_\alpha)\}_{\alpha\in A}$ where A is an indexing set which cover M in the sense that $M = \bigcup_{\alpha\in A} U$ and such that whenever $U_\alpha \cap U_\beta$ is not empty then the map $x_\beta \circ x_\alpha^{-1} : x_\alpha(U_\alpha \bigcap U_\beta) \to x_\beta(U_\alpha \bigcap U_\beta)$ is a member of the atlas. The maps $x_\beta \circ x_\alpha^{-1}$ are called "transition maps" or "coordinate change maps".

Definition 3.6 (Covering Space)

Let M and $\widetilde{M}$ be C^r spaces. A surjective C^r map $\rho : \widetilde{M} \to M$ is called a C^r covering map if every point $p \in M$ has an open connected neighborhood U such that each connected component $\widetilde{U}_i$ of $\rho^{-1}(U)$ is C^r diffeomorphic to U via the restriction $\rho|_{\widetilde{U}_i} : \widetilde{U}_i \to U$. We say that U is evenly covered. The triple $(\widetilde{M}, \rho, M)$ is called a covering space. We also refer to the space $\widetilde{M}$ as a covering space for M. $\widetilde{M}$ is covering manifold.



Definition 3.7 (Immersion)

A map $f : M \to N$ is called immersion at $p \in M$ iff $T_p f : T_p M \to T_{f(p)} N$ is a linear injection at *p*. A map $f : M \to N$ is called an immersion if *f* is an immersion at every $p \in M$. p is the perturbation point of the dynamic system.

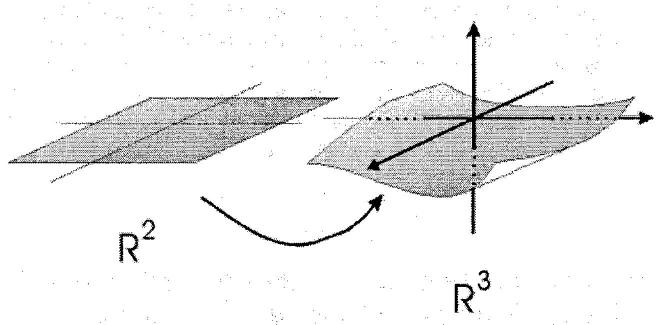
Theorem 1: Let $f : M^n \to N^d$ be a smooth function which is an immersion at ***p***. Then $f : M^n \to N^d$ there exists charts $x :: (M^n, p) \to (\Re^n, 0)$ and $y :: (N^d, f(p)) \to (\Re^d, 0)$ such that

$y \circ f \circ x^{-1} :: \Re^n \to \Re^n x \Re^{d-n}$

is given by $x \mapsto (x,0)$ near 0. In other words, there is open set $U \subset M$ such that f(U) is a submanifold of N the expression for f is $(x^1, x^2, \cdots x^n) \to (x^1, x^2, \cdots x^n, 0, \cdots, 0) \in \Re^d$

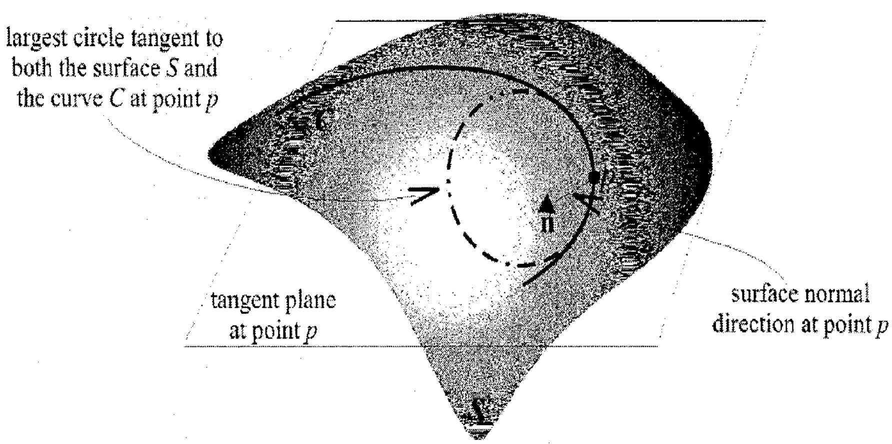
In our problem, we consider $y :: \dot{x}$ in our autonomous first order ODE set of dynamics.

Theorem 2: If $f : M \to N$ is an immersion (immersion at each point) and if f is a homeomorphism onto its image $f(M)$ using the relative topology, then $f(M)$ is a regular submanifold of N. In this case we call $f : M \to N$ an **embedding**.



Example:

A cerebral cortex can be represented on a manifold as shown in figure using topology [2]. The cortex is most complex dynamical system in Universe which shows phase transitions at mesoscopic level.



Definition 3.8 A graded derivation of degree l on a differential form $\Omega \to \Omega_M$ is a map $D:\Omega \to \Omega_M$ such that for each $U \subset M$. $\Omega_M(U)$ are differential forms on U manifold. k is the power of unstable manifold dimension.

$D:\Omega^k(U) \to \Omega^{k+l}(U)$ and such that for $\alpha \in \Omega^k(U)$ and $\beta \in \Omega(U)$ we have

$$D(\alpha \wedge \beta) = D\alpha \wedge \beta + (-1)^{kl} \alpha \wedge D\beta \quad (2.4)$$

Graded derivation is completely determined.

Theorem (Exterior Derivative for degree one map) There is a unique graded map $d:\Omega_M \to \Omega_M^{k+1}$ called the exterior derivative such that

1) $d \circ d = 0$

2) d is a graded derivation of degree one that is

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta) \textbf{ for } \alpha \in \Omega_M^k$$

4.An open problem in biological dynamical systems in terms of differentiable manifolds and covering maps

Let $A = \oplus_{k \in Z} A^k$ and $B = \oplus_{k \in Z} B^k$ be differential biological manifolds like (M,N). A map $g: A \to B$ is called a map if g is a degree 0 graded map such that $g_{k-1} \circ d \circ g_k^{-1} :: \Re^{k-1} \to \Re^{k-1} x \Re$. Melnikov function will discriminate whether chaotic motion can occur.

$$Mel(t_0) = \int_{-\infty}^{\infty} g_{k-1}[x_h(\zeta)] \wedge d[x_h(\zeta), \zeta + t_0] d\zeta$$

$$Mel(t_1) = \int_{-\infty}^{\infty} d[x_h(\zeta)] \wedge g_k^{-1}[x_h(\zeta), \zeta + t_1] d\zeta$$

If we obtain a $\{Mel(t_0), Mel(t_1), \cdots Mel(t_n)\}$ distance function of Cauchy series of form which is continuously decreasing function, we can call this system non-chaotic.

For example $\{t_0, t_1, \cdots t_k\}$ denote spike timings of one neuron in a two dimensional planar manifold.

Following commutative diagram satisfies a nonlinear dynamical system for all extended phase dimension of k.

$$\begin{array}{ccccccc} \xrightarrow{d} & A^{k-1} & \xrightarrow{d} & A^{k} & \xrightarrow{d} & A^{k+1} & \xrightarrow{d} \\ & \downarrow g_{k-1} & & \downarrow g_{k} & & \downarrow g_{k+1} & \\ \xrightarrow{d} & B^{k-1} & \xrightarrow{d} & B^{k} & \xrightarrow{d} & B^{k-1} & \xrightarrow{d} \end{array}$$

There are inherent maps in biological systems which cause to increase the order of biological system preventing system to show fast transition and decrease the chaotic behaviour.

$$f : d : A^{k-1} \rightarrow A^{k}$$

$$f : d : A^{k} \rightarrow A^{k+1}$$

We have two charts

$$g_{k-1} :: (A^{k-1}, p) \rightarrow (\Re^{k-1}, 0)$$

$$g_{k} :: (A^{k}, d(p)) \rightarrow (\Re^{k}, 0)$$

Such that

$$g_{k-1} \circ d \circ g_{k}^{-1} :: \Re^{k-1} \rightarrow \Re^{k-1} x \Re$$

5.Conclusion

In conclusion, a new approach is introduced to explore chaotic or non chaotic behavior of biological systems by looking at the dynamics making use of differential geometry was suggested. We suggested a new open problem whether chaos exits in nonlinear dynamical systems or not. If we associate algebraic topology with time dynamics systems we can explain transitions of a trajectory among stable and unstable manifolds for a biological system and will lead us to change our views to nonlinear dynamical systems. We need to questionnaire us when a bifurcation occurs, the system is suspended (embedded) it in the extended phase space by making transitions more smooth in overlapped manifolds in different trajectory. We need to look at both topological dynamics and time dynamics of a biological system when analyzing systems in nature. In future we will show cohomology equivalency of classes of manifolds on a ring structure and defining cup product in a dynamics.

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