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COALGEBRAIC MODAL LOGIC for $\mathcal{P}_\omega$

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ABSTRACT

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Modal logic plays an important role in many areas of computer science. In recent years coalgebras and their applications to computer science have attracted a lot of attention because coalgebras have been introduced to model various types of transition systems. In this thesis we study $\mathcal{P}_\omega$-coalgebras and coalgebraic modal logic corresponding to this functor. This thesis begins with some preliminary definitions, examples and propositions about modal logic and category theory. After the notion of coalgebra is introduced, some basic definitions, properties and examples about the subject is given. Then, the concept of predicate lifting is widely mentioned. Next, some propositions and theorems are proven on predicate liftings. Finally, the coalgebraic modal logic corresponding to the finite power set functor is defined.

Keywords: Modal logic, Coalgebra, Bisimulation, Predicate lifting

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ÖZET

$\mathcal{P}_w$ için Kocebirsel Modal Mantık

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Modal mantık bilgisayar biliminin pek çok alanında önemli bir yer tutmaktadır. Son yıllarda kocebirler ve bunların bilgisayar bilmine uygulamaları ilgi çekmektedir çünkü kocebirlerin çeşitli aktarım sistemlerini modelledikleri gösterilmiştir. Evrensel cebir teoerisine dual olarak gelişen kocebir teorisi doğal olarak kategori teoriye dayalıdır. Bu nedenle bu tezde öncelikle modal mantıktan ve kategori teoriden temel bilgiler verilmiştir. Sonra kocebirler tanıtılarak bunların temel özellikleri ile bu özelliklerin ispatları ve konunun temel örnekleri verilmiştir. Son olarak modal operatörlerin yorumlanmasına olanak sağlayan doğal dönüşümler tanıtılmış, özelikleri ispatlanmış ve sonlu kuvvet funktoru $\mathcal{P}_\omega$' ya tekabül eden kocebirsel mantık tanımlanmıstır.

Anahtar Kelimeler: Modal mantık, kocebir, bisimilasyon, doğal dönüşüm

to my dear family...

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Chapter 1

Introduction

Modal logic plays an important role in many areas of computer science. In recent years coalgebras and their applications to computer science have received a lot of attention [?]. Coalgebras have been introduced by Aczel and Mendler [?] to model various types of transition systems in 1989. Rutten [?] develops the fundamental theory of universal coalgebra along the lines of universal algebra. In the semantics of programming, data types are usually presented as algebras. For instance, the collection of finite words A^* over some alphabet A is an algebra. For infinite data structures, the dual notion of coalgebra has been used as an alternative to the algebraic approach. In universal algebra central concepts are Σ algebra, homomorphisms of Σ-algebras and congruence relation. The corresponding notions on coalgebra are coalgebra morphisms of coalgebras, and bisimulation equivalence.

Coalgebras generalize relational structures and coalgebraic modal logic is a generalization of a basic propositional modal logic and allows us to reason about states of coalgebras for an endofunctor on the category of sets [?]. As is mentioned in [?]: Which modal languages can be used for reasoning about coalgebras is an issue which is still under discussion. Research on this question goes back to work by Moss. Moss' coalgebraic logic assigns a logical language to every weak pullback preserving endofunctor on the category of sets. His syntax allows for infinite conjunctions and contains a somewhat non-standard modal operator. Moss' coalgebraic logic was followed by Kurz and Rossiger. Kurz defined a finitary, multi-modal language for coalgebras for a limited class of endofunctors on Set. In 2006 Kupke [?] have used a finitary syntax for specifying coalgebras on modal languages. They have observed that coalgebras for functors on the category of Stone-spaces have been a useful tool in studying coalgebraic modal logics and constructed Stone coalgebras.

Another line of research in the area of coalgebraic modal logic started with the work of Pattinson [?]. The logics are studied within the abstract framework of coalgebraic modal logic, which can be instantiated with arbitrary endofunctors on the category of sets. This is achieved through the use of predicate liftings, which generalize atomic propositions and modal operators from Kripke models to arbitrary coalgebras [?].

The aim of this thesis is to present the coalgebraic modal logic corresponding to the finite power set functor using predicate liftings. We have followed Pattinson's approach.

Chapter 2

Preliminaries

2.1 Modal Languages

In this section we give basic definitions. These definitions can be found in [?].

Definition 2.1.1 The *basic modal language* is defined using a set of propositional letters Φ whose elements are usually denoted p, q, r and so on, and a unary modal operator $\Diamond$ ('*diamond*').

The well-formed formulas φ of the basic modal language are given by the rule

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \psi \vee \varphi \mid \Diamond\varphi,$$

where p ranges over elements of Φ.

We have a dual operator $\Box$ ('*box*') for our diamond which is defined by

$$\Box\phi := \neg\Diamond\neg\phi.$$

We assume that the set Φ of propositional letters is a countably infinite set.

Definition 2.1.2 A modal similarity type is a pair $\tau = (O, P)$ where O is a non-empty set, and ρ is a function $O \longrightarrow \mathbf{N}$. The elements of O are called *modal operators*.

.

2.2 Frames and Models

We define frames and models for the basic modal language.

Definition 2.2.1 A *Kripke frame* for the basic modal language is a pair $\mathcal{F} = (W, R)$ such that

(i) the non-empty set W is the carrier of the frame and its elements are called *states* or *worlds*.

(ii) R is a binary relation on W.

Frames are *relational structures*.

Definition 2.2.2 A *model* for the basic modal language is a pair $\mathcal{M} = (\mathcal{F}, V)$, where $\mathcal{F}$ is a frame for the basic modal language, and V is a function assigning to each propositional letter p in Φ a subset $V(p)$ of W. Formally, $V : \Phi \longrightarrow \mathcal{P}(W)$ is a map, where $\mathcal{P}(W)$ denotes the power set of W. Informally, we think of $V(p)$ as the set of points in our model where p is true. The function V is called a *valuation.* Given a model $\mathcal{M} = (\mathcal{F}, V)$, we say that $\mathcal{M}$ is *based on* the frame $\mathcal{F}$.

Definition 2.2.3 (Validity in a frame)

(i) A formula ϕ is valid at a state w in a frame $\mathcal{F}$ (notation: $\mathcal{F}, w \Vdash \phi$) if ϕ is true at w in every model $(\mathcal{F}, V)$ based on $\mathcal{F}$;

(ii) ϕ is valid in a frame $\mathcal{F}$ (notation: $\mathcal{F} \Vdash \phi$) if it is valid at every state in $\mathcal{F}$.

Validity differs from truth in many ways. For example, when a formula $\phi \vee \psi$ is true at a point w, this means that either ϕ or ψ is true at w. On the other hand, if $\phi \vee \psi$ is valid on a frame $\mathcal{F}$, this does not mean that either ϕ or ω is valid on $\mathcal{F}$. $p \vee \neg p$ is a simple counter example.

Example 2.2.4 The formula $\Diamond(p \vee q) \longrightarrow (\Diamond p \vee \Diamond q)$ is valid on all frames. To see this, take any frame $\mathcal{F}$ and state w in $\mathcal{F}$. Let V be a valuation on $\mathcal{F}$. We have to show that if $(\mathcal{F}, V), w \Vdash \Diamond(p \vee q)$, then $(\mathcal{F}, V), w \Vdash (\Diamond p \vee \Diamond q)$. So, assume that $(\mathcal{F}, V), w \Vdash \Diamond(p \vee q)$. Then, by definition there is a state v such that Rwv and $(\mathcal{F}, V), v \Vdash p \vee q$ but if $v \Vdash p \vee q$ then either $v \Vdash p$ or $v \Vdash q$. Hence either $w \Vdash \Diamond p$ or $w \Vdash \Diamond q$. Either way, $w \Vdash \Diamond p \vee \Diamond q$.

Example 2.2.5 The formula $\Diamond\Diamond p \longrightarrow \Diamond p$ is not valid on all frames. To see this we need to find a frame $\mathcal{F}$, a state w in $\mathcal{F}$ and a valuation V on $\mathcal{F}$ that falsifies the formula at w. So, let $\mathcal{F}$ be a three-point frame with universe $\{0, 1, 2\}$ and relation $\{(0, 1),\ (1, 2)\}$ .Let V be any valuation on $\mathcal{F}$ such that $V(p) = \{2\}$. Then $(\mathcal{F}, V), 0 \Vdash \Diamond\Diamond p$, but $(\mathcal{F}, V), 0 \nVdash \Diamond p$ since 0 is not related to 2. On the other hand, there is a class of frames on which $\Diamond\Diamond p \longrightarrow \Diamond p$ is valid: the class of transitive frames. To see this, take any transitive frame $\mathcal{F}$ and a state w in $\mathcal{F}$. Let V be a valuation on $\mathcal{F}$. We have to show that if $(\mathcal{F}, V), w \Vdash \Diamond\Diamond p$, then also $(\mathcal{F}, V), w \Vdash \Diamond p$. So assume that $(\mathcal{F}, V), w \Vdash \Diamond\Diamond p$. Then by definition there are states u and v such that Rwu and Ruv with $(\mathcal{F}, V), v \Vdash p$ but as R is transitive, it follows that Rwv, hence $(\mathcal{F}, V), w \Vdash \Diamond p$.

Now, we see how to interpret the basic modal language in models by means of the following satisfaction definition.

Definition 2.2.6 (Satisfaction in a model) Suppose w is a state in a model $\mathcal{M} = (W, R, V)$. Then, we inductively define the notion of a formula ϕ being *satisfied (or true)* in $\mathcal{M}$ at state w as follows:

$$
\begin{array}{lll}
\mathcal{M}, w \Vdash p & \text{iff} & w \in V(p),\ \text{where } p \in \Phi, \\
\mathcal{M}, w \Vdash \bot & & \text{never}, \\
\mathcal{M}, w \Vdash \neg\phi & \text{iff} & \text{not } \mathcal{M}, w \Vdash \phi, \\
\mathcal{M}, w \Vdash \phi \vee \psi & \text{iff} & \mathcal{M}, w \Vdash \phi \text{ or } \mathcal{M}, w \Vdash \psi, \\
\mathcal{M}, w \Vdash \Diamond\phi & \text{iff} & \text{for some } v \in W \ \text{ with } Rwv \text{ we have } \mathcal{M}, v \Vdash \phi.
\end{array}
$$

It follows from this definition that $\mathcal{M}, w \Vdash \Box\phi$ iff for all $v \in W$ such that Rwv, we have $\mathcal{M}, v \Vdash \phi$.

If $\mathcal{M}$ does not satisfy ϕ at w we often write $\mathcal{M}, w \nVdash \phi$, and say that ϕ is false at w. It is convenient to extend the valuation V from propositional letters to arbitrary formulas so that $V(\phi)$ always denotes the set of states at which ϕ is true.

$$V(\phi) := \{w \mid \mathcal{M}, w \Vdash \phi\}.$$

Example 2.2.7 Consider the frame $\mathcal{F} = (\{w_1, w_2, w_3, w_4, w_5\}, R)$, where Rw_iw_j iff $j = i + 1$:

$$w_1 \longrightarrow w_2 \longrightarrow w_3 \longrightarrow w_4 \longrightarrow w_5$$

If we choose the valuation V on $\mathcal{F}$ such that $V(p) = \{w_2, w_3\}, V(q) = \{w_1, w_2, w_3, w_4, w_5\}$, and $V(r) = \emptyset$, then in a model $\mathcal{M} = (\mathcal{F}, V)$ we have that

- $\mathcal{M}, w_1 \Vdash \Diamond\Box p$ because Rw_1w_2 and $\mathcal{M}, w_2 \Vdash \Box p$ since $\mathcal{M}, w_3 \Vdash p$.
- $\mathcal{M}, w_1 \nVdash \Diamond\Box p \longrightarrow p$ because while $\mathcal{M}, w_1 \Vdash \Diamond\Box p$, $\mathcal{M}, w_1 \nVdash p$.
- $\mathcal{M}, w_2 \Vdash \Diamond(p \land \neg r)$ because Rw_2w_3 and $\mathcal{M}, w_3 \Vdash p$ and $\mathcal{M}, w_3 \nVdash r$.

- Since

$$
\begin{aligned}
\mathcal{M}, w_1 \Vdash q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond q))) \quad \text{iff} \quad & \mathcal{M}, w_1 \Vdash q \text{ and} \\
& \mathcal{M}, w_1 \Vdash \Diamond(q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond q))) \\
\text{iff} \quad & \mathcal{M}, w_1 \Vdash q \text{ and } (Rw_1w_2 \text{ and} \\
& \mathcal{M}, w_2 \Vdash q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond q))) \\
\text{iff} \quad & \mathcal{M}, w_1 \Vdash q \text{ and } (Rw_1w_2 \text{ and} \\
& (\mathcal{M}, w_2 \Vdash q \text{ and} \\
& \mathcal{M}, w_2 \Vdash \Diamond(q \wedge \Diamond(q \wedge \Diamond q)))) \\
\text{iff} \quad & \mathcal{M}, w_1 \Vdash q \text{ and } (Rw_1w_2 \text{ and} \\
& (\mathcal{M}, w_2 \Vdash q \text{ and } (Rw_2w_3 \text{ and} \\
& (\mathcal{M}, w_3 \Vdash q \text{ and } \mathcal{M}, w_3 \Vdash \Diamond(q \wedge \Diamond q))))) \\
\text{iff} \quad & \mathcal{M}, w_1 \Vdash q \text{ and } (Rw_1w_2 \text{ and} \\
& (\mathcal{M}, w_2 \Vdash q \text{ and } (Rw_2w_3 \text{ and} \\
& (\mathcal{M}, w_3 \Vdash q \text{ and } (Rw_3w_4 \text{ and} \\
& \mathcal{M}, w_4 \Vdash q \text{ and } \mathcal{M}, w_4 \Vdash \Diamond q))))) \\
\text{iff} \quad & \mathcal{M}, w_1 \Vdash q \text{ and } (Rw_1w_2 \text{ and} \\
& (\mathcal{M}, w_2 \Vdash q \text{ and } (Rw_2w_3 \text{ and} \\
& (\mathcal{M}, w_3 \Vdash q \text{ and } (Rw_3w_4 \text{ and} \\
& (\mathcal{M}, w_4 \Vdash q \text{ and } (Rw_4w_5 \text{ and} \\
& \mathcal{M}, w_5 \Vdash q))))))),
\end{aligned}
$$

$\mathcal{M}, w_1 \Vdash q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond(q \wedge \Diamond q)))$.

Furthermore, $\mathcal{M} \Vdash \Box q$ because $\mathcal{M}, w_i \Vdash q \ \forall i = 1, 2, 3, 4, 5$.

Definition 2.2.9 (Bounded Morphism) A mapping $f : \mathcal{M} = (W, R, V) \longrightarrow \mathcal{M}' = (W', R', V')$ is a bounded morphism if it satisfies the following conditions:

(i) w and $f(w)$ satisfy the same propositional letters.

(ii) f is a homomorphism with respect to the relation R that is, if Rwv then $R'f(w)f(v)$.

(iii) If $R'f(w)v'$ then there exists v such that Rwv and $f(v) = v'$ (the back condition).

Example 2.2.10 Consider the models $\mathcal{M} = (W, R, V)$ and $\mathcal{M}^{'} = (W^{'}, R^{'}, V^{'})$ where

- $W = \mathbf{N}$ (the natural numbers), Rmn iff $n = m+1$ and $V(p) = \{n \in \mid n \; is \; even\}$,
- $W^{'} = \{e, o\}$, $R^{'} = \{(e, o), (o, e)\}$ and $V^{'}(p) = \{e\}$.

Now, let f be the following map:

$$
\begin{array}{rcll}
f: \; W & \longrightarrow & W^{'} & \\
n & \longmapsto & f(n) = \{ & \begin{array}{l} e \; if \; n \; is \; even \\ o \; if \; n \; is \; odd \end{array}
\end{array}
$$

We claim that f is a bounded morphism from $\mathcal{M}$ to $\mathcal{M}^{'}$. Trivially, f satisfies item (i) of the definition because for any even natural number n, $f(n) = e$ and they satisfy the same propositional letter p. On the other hand, for any odd natural number n, $f(n) = o$ and there is no propositional letter satisfied in both n and o. So, we can deduce that for all $n \in \mathbf{N}$, n and $f(n)$ satisfy the same propositional letters. As for the homomorphic condition consider an arbitrary pair $(n, n+1)$ in R. There are two possibilities: n is either even or odd. Suppose n is even. Then $n + 1$ is odd, so $f(n) = e$ and $f(n + 1) = o$ but then we have $R^{'}eo$.Similarly, if n is odd then $n + 1$ is even. Hence we have $f(n) = o$ and $f(n + 1) = e$ but then $(o, e) \in R^{'}$. As a result, for all pairs $(n, n + 1) \in R$, $R^{'}f(n)f(n + 1)$. Now for the interesting part: the back condition. Take an arbitrary element n of W and assume that $R^{'}f(n)w^{'}$. We have to find an $m \in \mathbf{N}$ such that Rnm and $f(m) = w^{'}$. Let us suppose that n is odd. As n is odd, $f(n) = o$, so by definition of $R^{'}$, we must have that $w^{'} = e$ but then $f(n + 1) = w^{'}$ since $n + 1$ is even. By the definition of R, we have that $n + 1$ is a successor of n. Hence, $n + 1$ is the m we are looking for. On the other hand, if n is even then $f(n) = e$ but then $w^{'} = o$. Since $n + 1$ is odd, $(n, n + 1) \in R$ and $f(n + 1) = o$, here $n + 1$ is the m we are searching for. Hence, back condition also holds.

2.3 Bisimulations

In this section, we introduce the concept of bisimulation. Bisimulations reflect the locality of the modal satisfaction relation. A bisimulation is a relation between two models in which related states have identical atomic information and matching transition possibilities.

Definition 2.3.1 Let $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be two models. A non-empty binary relation $Z \subseteq W \times W'$ is called a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ (notation: $Z : \mathcal{M} \underline{\leftrightarrow} \mathcal{M}'$) if the following conditions are satisfied:

(i) If wZw' then w and w' satisfy the same propositional letters.

(ii) If wZw' and Rwv then there exists v' in $\mathcal{M}'$ such that vZv' and $R'w'v'$ (the forth condition).

(iii) The converse of (ii) : If wZw' and $R'w'v'$ then there exists v in $\mathcal{M}$ such that vZv' and Rwv (the back condition).

When Z is a bisimulation linking two states w in $\mathcal{M}$ and w' in $\mathcal{M}'$ we say that w and $w'$are bisimilar and we write $Z : \mathcal{M}, w \underline{\leftrightarrow} \mathcal{M}', w'$.

Example 2.3.2 Let the models $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be as they are shown in the following figure. Let $V(p) = \{1, 3\}$, $V(q) = \{2, 4, 5\}$ and $V'(p) = \{a, d\}$, $V'(q) = \{b, c, e\}$.

$$\begin{array}{cccccccccccccccc}
 & & & & & & & 4 & & & b & & & & & \\
 & & & & & & \nearrow & & & \nearrow & & \searrow & & & & \\
\mathcal{M}: & 1 & \longrightarrow & 2 & \longrightarrow & 3 & & & a & & & & d & \longrightarrow & e & : \ \mathcal{M}' \\
 & & & & & & \searrow & & & \searrow & & \nearrow & & & & \\
 & & & & & & & 5 & & & c & & & & &
\end{array}$$

We claim that $\mathcal{M}$ and $\mathcal{M}'$ are bisimilar. To see this, define the following relation Z between their states: $Z = \{(1, a),\ (2, b),\ (2, c),\ (3, d),\ (4, e),\ (5, e)\}$. For any

$(x, y) \in Z$, x and y make the same propositional variables true. Hence, condition (i) of the definition is obviously satisfied.

As for forth condition, let's check whether any move in $\mathcal{M}$ is matched by a similar move in $\mathcal{M}'$.

- For $(1, a) \in Z$ and $(1, 2) \in R$, $\exists b \in W'$ such that $(a, b) \in R'$ and $(2, b) \in Z$.
- For $(2, b) \in Z$ and $(2, 3) \in R$, $\exists d \in W'$ such that $(b, d) \in R'$ and $(3, d) \in Z$.
- For $(2, c) \in Z$ and $(2, 3) \in R$, $\exists d \in W'$ such that $(c, d) \in R'$ and $(3, d) \in Z$.
- For $(3, d) \in Z$ and $(3, 4) \in R$, $\exists e \in W'$ such that $(d, e) \in R'$ and $(4, e) \in Z$.
- For $(3, d) \in Z$ and $(3, 5) \in R$, $\exists e \in W'$ such that $(d, e) \in R'$ and $(5, e) \in Z$.

So, forth condition holds. Now, let us show that the back condition is also satisfied.

- For $(1, a) \in Z$ and $(a, c) \in R'$, $\exists 2 \in W$ such that $(1, 2) \in R$ and $(2, c) \in Z$.
- For $(1, a) \in Z$ and $(a, b) \in R'$, $\exists 2 \in W$ such that $(1, 2) \in R$ and $(2, b) \in Z$.
- For $(2, b) \in Z$ and $(b, d) \in R'$, $\exists 3 \in W$ such that $(2, 3) \in R$ and $(3, d) \in Z$.
- For $(2, c) \in Z$ and $(c, d) \in R'$, $\exists 3 \in W$ such that $(2, 3) \in R$ and $(3, d) \in Z$.
- For $(3, d) \in Z$ and $(d, e) \in R'$, $\exists 4 \in W$ such that $(3, 4) \in R$ and $(4, e) \in Z$.

Therefore, Z is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$.

Chapter 3

Coalgebras

Coalgebras are category theoretic notions and they generalize the relational structures. Every coalgebra is based on a *carrier* which is an object in the so-called base category and any functor F on a category gives rise to F-coalgebras. Because of this fact we start to this chapter giving basic categorical concepts.

3.1 Basic Notions of Category Theory

We first study axioms for a category and some structures in it which we will use in this thesis. These can be found in [?] or [?].

Definition 3.1.1 A *category* $\mathbf{A}$ consists of

(1) a class of *objects* $A, B, C, ...,$

(2) a class of *morphisms* or *arrows* $f, g, h, ...$between these objects,

(3) for each $\mathbf{A}$-object A, a morphism $1_A : A \longrightarrow A$, called the $\mathbf{A}$-identity on A and

(4) a composition law associating with each $\mathbf{A}$-morphism $f : A \longrightarrow B$ and each $\mathbf{A}$-morphism $g : B \longrightarrow C$, an $\mathbf{A}$-morphism $g \circ f : A \longrightarrow C$, called the composite of f and g subject to the following conditions:

(a) for every composable pair f and g the composite $f \circ g$ goes from the domain of g to the codomain of f,

(b) composition is associative; i.e., for morphisms $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : C \longrightarrow D$, the equation $h \circ (g \circ f) = (h \circ g) \circ f$ and

(c) for **A**-morphisms $f : A \longrightarrow B$, we have $1_B \circ f = f$ and $f \circ 1_A = f$.

Example 3.1.2 The category **Set** whose objects are all sets and whose morphisms are all mappings between sets, forms a standard example of a category.

Example 3.1.3 The category **Grp** with objects all groups and arrows all homomorphisms between them, constitutes another example of a category.

Definition 3.1.4 If **A** and **B** are categories, then a *functor* F from **A** to **B**, written $F : \mathbf{A} \longrightarrow \mathbf{B}$, is a map that assigns to each object A of **A** an object FA of **B** and each arrow f of **A** an arrow Ff of **B**, meeting the following conditions:

(1) It preserves domains and codomains: given $f : A \longrightarrow B$ of **A**, we have $Ff : FA \longrightarrow FB$ in **B**.

(2) It preserves identities: for any A of **A**, $F(1_A) = 1_{FA}$.

(3) It preserves compositions: if f and g are composable in **A** then $F(g \circ f) = Fg \circ Ff$, where the second composite is formed in **B**.

Since a functor $F : \mathbf{A} \longrightarrow \mathbf{B}$ preserves composition and identities, it preserves inverses; that is, if $g = f^{-1}$ in **A** then $Fg = (Ff)^{-1}$ in **B**.

We define the composition of two functors $F : \mathbf{A} \longrightarrow \mathbf{B}$ and $G : \mathbf{B} \longrightarrow \mathbf{C}$, denoted by the composite $G \circ F : \mathbf{A} \longrightarrow \mathbf{C}$ with the following rules:

(1)$(G \circ F)(A) = G(F(A))$ for any object A in **A** and

(2)$(G \circ F)(f) = G(F(f))$ for any arrow f in **A**.

Every category **A** has a *dual category* $\mathbf{A}^{op}$. The objects of $\mathbf{A}^{op}$ are the objects of **A** and the arrows of $\mathbf{A}^{op}$ are the arrows of **A** but domain and codomain are reversed. So every category is the dual of its dual: $\mathbf{A} = (\mathbf{A}^{op})^{op}$.

A functor $G : \mathbf{A}^{op} \longrightarrow \mathbf{B}$ is often a *contravariant functor* from $\mathbf{A}$ to $\mathbf{B}$. That is, G assigns to each object A of $\mathbf{A}$ an object GA of $\mathbf{B}$ and to each $f : A \longrightarrow A'$ of $\mathbf{A}$, an arrow $Gf : GA' \longrightarrow GA$ satisfying the conditions $G(1_{\mathbf{A}}) = 1_{G\mathbf{A}}$ and $G(g \circ f) = (Gf) \circ (Gg)$.

Example 3.1.5

(i) **The identity functor:** The identity functor $I : \mathbf{Set} \longrightarrow \mathbf{Set}$ sends sets and functions to themselves.

(ii) **The constant functor:** The constant functor $A : \mathbf{Set} \longrightarrow \mathbf{Set}$, where A is any set, maps any set to the set A and any function to the identity function 1_A on A.

(iii) **The power set functor:** The power set functor $\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set}$ maps a set S to the set of all its subsets $\mathcal{P}(S) = \{V \mid V \subseteq S\}$ and a function $f : S \longrightarrow T$ to $\mathcal{P}(f) : \mathcal{P}(S) \longrightarrow \mathcal{P}(T)$, which is defined, for any $V \subseteq S$, by $\mathcal{P}(f)(V) = f[V] = \{f(u) \mid u \in V\}$.

(iv)**The finite power set functor:** The finite power set functor $\mathcal{P}_\omega : \mathbf{Set} \longrightarrow \mathbf{Set}$ maps a set S to the set of all its finite subsets

$$\mathcal{P}_\omega(S) = \{V \mid V \subseteq S \text{ and } V \text{ is finite}\}$$

and a function $h : S \longrightarrow T$ to $\mathcal{P}_\omega(h) : \mathcal{P}_\omega(S) \longrightarrow \mathcal{P}_\omega(T)$ which is defined, for any $V \in \mathcal{P}_\omega(S)$, by $\mathcal{P}_\omega(h)(V) = f[V]$.

(v) **The contravariant power set functor:** The contravariant power set functor $\bar{\mathcal{P}} : \mathbf{Set}^{op} \longrightarrow \mathbf{Set}$ acts on set as $\mathcal{P}$ does: for any set S, $\bar{\mathcal{P}}(S) = \mathcal{P}(S)$. A function $f : S \longrightarrow T$ is mapped to $\bar{\mathcal{P}}(f) : \bar{\mathcal{P}}(T) \longrightarrow \bar{\mathcal{P}}(S)$, which is defined for any $V \subseteq T$, by $\bar{\mathcal{P}}(f)(V) = f^{-1}[V] = \{x \in S \mid f(x) \in V\}$.

The contravariant power set functor will be considered in composition with itself:

$$\begin{array}{lcl} \mathbf{Set} & \xrightarrow{\quad \bar{\mathcal{P}} \circ \bar{\mathcal{P}} \quad} & \mathbf{Set} \\ & & \\ S & & (\bar{\mathcal{P}} \circ \bar{\mathcal{P}})(S) \\ \downarrow f & & \quad \downarrow (\bar{\mathcal{P}} \circ \bar{\mathcal{P}})(f) \\ T & & (\bar{\mathcal{P}} \circ \bar{\mathcal{P}})(T) \end{array}$$

For any $f : S \longrightarrow T$, $(\bar{\mathcal{P}} \circ \bar{\mathcal{P}})(f) : (\bar{\mathcal{P}} \circ \bar{\mathcal{P}})(S) \longrightarrow (\bar{\mathcal{P}} \circ \bar{\mathcal{P}})(T)$ is defined as for any $V \subseteq \mathcal{P}\ (S)$, $(\bar{\mathcal{P}} \circ \bar{\mathcal{P}})(f)(V) = \{W \subseteq T \mid f^{-1}[W] \in V\}$. We can verify this as follows:

$$\begin{array}{lclclcl} \mathbf{Set}^{op} & \xrightarrow{\quad \bar{\mathcal{P}} \quad} & \mathbf{Set}^{op} & \xrightarrow{\quad \bar{\mathcal{P}} \quad} & \mathbf{Set} \\ & & & & \\ S & & (\bar{\mathcal{P}})(S) = \mathcal{P}(S) & & (\bar{\mathcal{P}} \circ \bar{\mathcal{P}})(S) \\ \downarrow f & & \uparrow (\bar{\mathcal{P}})(f) & & \downarrow (\bar{\mathcal{P}} \circ \bar{\mathcal{P}})(f) \\ T & & (\bar{\mathcal{P}})(T) = \mathcal{P}(T) & & (\bar{\mathcal{P}} \circ \bar{\mathcal{P}})(T) \end{array}$$

For any $V \in (\bar{\mathcal{P}} \circ \bar{\mathcal{P}})(S)$,

$$\begin{aligned} (\bar{\mathcal{P}} \circ \bar{\mathcal{P}})(f)(V) &= \bar{\mathcal{P}}(\bar{\mathcal{P}}(f))(V) \\ &= (\bar{\mathcal{P}}(f))^{-1}[V] \\ &= \left\{W \in \mathcal{P}(T) \mid \bar{\mathcal{P}}(f)(W) \in V\right\} \\ &= \{W \subseteq T \mid f^{-1}[W] \in V\}. \end{aligned}$$

3.2 Pullbacks

The following definitions, examples and propositions can be found in [ML] and [J].

Definition 3.2.1 A *pullback* of functions $f : S \longrightarrow T$ and $g : U \longrightarrow T$ is a triple $(P, k : P \longrightarrow S, \ell : P \longrightarrow U)$ with $f \circ k = g \circ \ell$ such that for any set X and functions $i : X \longrightarrow S$ and $j : X \longrightarrow U$ satisfying $f \circ i = g \circ j$ there exists a unique (so-called mediating) function $u : X \longrightarrow P$ that makes the whole diagram commute i.e. $k \circ u = i$ and $\ell \circ u = j$.

In this thesis, we denote it by $pb(f, g) = (P, k, \ell)$.

Example 3.2.2 In **Set**, a pullback of any corner of arrows $f : S \longrightarrow T$ and $g : U \longrightarrow T$ always exist, given by the set

$$P = \{\langle s, u\rangle \in S \times U \mid f(s) = g(u)\},$$

together with projections $\pi_1 : P \longrightarrow S$ and $\pi_2 : P \longrightarrow U$.

A weak pullback is defined in the same way as a pullback, but without the requirement that the *mediating* function be unique. We will also denote a weak pullback of two arrows $f : S \longrightarrow T$ and $g : U \longrightarrow T$ by $wpb(f, g) = (P, k, \ell)$.

Hence, every pullback is a weak pullback but converse is, of course, not true in general. However, weak and ordinary pullbacks coincide if all functions involved are mono.

The requirement that functors preserve weak pullbacks is needed at various places in the Coalgebra Theory. Therefore, it is worthwhile to examine some of the functors that have this property. First, let us give the following definition and the proposition.

Definition 3.2.3 Let $T\colon \mathbf{C} \longrightarrow \mathbf{C}$ be a functor. We say that T preserves (weak) pullbacks if T transforms every (weak) pullback (P, π_1, π_2) of $f : A \longrightarrow C$ and $g : B \longrightarrow C$ into a (weak) pullback $(T(P), T(\pi_1), T(\pi_2))$ of $T(f)$ and $T(g)$.

Proposition 3.2.4 If a functor $F : \mathbf{Set} \longrightarrow \mathbf{Set}$ preserves pullbacks then it also preserves weak pullbacks.

Example 3.2.5

(i) The identity functor, $I : \mathbf{Set} \longrightarrow \mathbf{Set}$ preserves pullbacks. Let us verify this fact.

$$\textbf{Set} \xrightarrow{\quad I(\_)\quad} \textbf{Set}$$

$$\begin{array}{ccc} P & \xrightarrow{p_2} & C \\ p_1 \downarrow & & \downarrow g \\ B & \xrightarrow[f]{} & D \end{array} \qquad \begin{array}{ccc} I(P) = P & \xrightarrow{I(p_2) = p_2} & C = I(C) \\ I(p_1) = p_1 \downarrow & & \downarrow I(g) = g \\ I(B) = B & \xrightarrow[I(f) = f]{} & D = I(D) \end{array}$$

Let (P, p_1, p_2) be a pullback for f and g.Since I transforms this pullback diagram to itself as shown above, $(I(P), I(p_1)I(p_2))$ is also a pullback diagram for $I(f)$ and $I(g)$. As a result, I preserves pullbacks and so by the above proposition, also weak pullbacks.

(ii) Constant functors preserve pullbacks.

$$\textbf{Set} \xrightarrow{\quad F\quad} \textbf{Set}$$

$$\begin{array}{ccc} P & \xrightarrow{p_2} & C \\ p_1 \downarrow & & \downarrow g \\ B & \xrightarrow[f]{} & D \end{array} \qquad \begin{array}{ccc} F(P) = A & \xrightarrow{F(p_2) = 1_A} & A = F(C) \\ F(p_1) = 1_A \downarrow & & \downarrow 1_A = F(g) \\ F(B) = A & \xrightarrow[F(f) = 1_A]{} & A = F(D) \end{array}$$

Let $(P, p_1, p_2) = pb(f, g)$ and F be the constant functor on **Set** which maps any any set to A and any function to the identity function 1_A on A.Then, we have $F(f) \circ F(p_1) = 1_A = F(g) \circ F(p_2)$.As for universality condition, for (X, h, k) let $F(f) \circ h = F(g) \circ k$ but $F(f) = F(g) = 1_A$, so $h = k$.Thus, $h = k : X \longrightarrow F(P)$ is the unique arrow satisfying universality property. As a result, F preserves pullbacks and hence weak pullbacks.

(iii) The functor $(\_) \times A$ preserves pullbacks.

$$\mathbf{Set} \xrightarrow{\quad (\_) \times A \quad} \mathbf{Set}$$

$$\begin{array}{ccc} P & \xrightarrow{q_2} & C \\ q_1 \downarrow & & \downarrow g \\ B & \xrightarrow[f]{} & D \end{array} \qquad\qquad \begin{array}{ccc} P \times A & \xrightarrow{q_2 \times 1_A} & C \times A \\ q_1 \times 1_A \downarrow & & \downarrow g \times 1_A \\ B \times A & \xrightarrow[f \times 1_A]{} & D \times A \end{array}$$

Assume that $(P, q_1, q_2) = pb(f, g)$ then, by definition, $f \circ q_1 = g \circ q_2$.(*)

Now, we claim that $(P \times A, q_1 \times 1_A, q_2 \times 1_A)$ is a pullback for $f \times 1_A$ and $g \times 1_A$.

$$\begin{array}{ll} (f \times 1_A) \circ (q_1 \times 1_A) & = (f \circ q_1) \times 1_A \\ (by\ (*)) & = (g \circ q_2) \times 1_A \\ & = (g \times 1_A) \circ (q_2 \times 1_A) \end{array}$$

As for the universality property, for $(X, h : X \longrightarrow B \times A, k : X \longrightarrow C \times A)$, let $(f \times 1_A) \circ h = (g \times 1_A) \circ k$.(**) Now, consider the following diagrams:

$$\begin{array}{ccccc} & & X & & \\ p_1 \circ h \swarrow & & \downarrow h & & \searrow p_2 \circ h \\ & B \xleftarrow{p_1} & B \times A & \xrightarrow{p_2} & A \\ & f \downarrow & \downarrow f \times 1_A & & \downarrow 1_A \\ & D \xleftarrow{i_1} & D \times A & \xrightarrow{i_2} & A \\ & g \uparrow & \uparrow & & \uparrow 1_A \\ & C \xleftarrow{j_1} & C \times A & \xrightarrow{j_2} & A \\ j_1 \circ k \nwarrow & & \uparrow & \nearrow & \\ & & X & & \end{array}$$

$$\begin{array}{ccccc} & & X & & \\ \mu = f \circ (p_1 \circ h) \swarrow & & \downarrow \langle \mu, \nu \rangle & & \searrow p_2 \circ h = \nu \\ & D \xleftarrow[i_1]{} & D \times A & \xrightarrow[i_2]{} & A \end{array}$$

$$\begin{array}{ccccc} & & X & & \\ \varsigma = g \circ (j_1 \circ k) \swarrow & & \downarrow \langle \varsigma, \tau \rangle & & \searrow j_2 \circ k = \tau \\ & D \xleftarrow[\kappa_1]{} & D \times A & \xrightarrow[\kappa_2]{} & A \end{array}$$

Hence, by uniqueness in the definition of a pullback diagram $(f \times 1_A) \circ h = \langle f \circ (p_1 \circ h), p_2 \circ h \rangle$ and $(g \times 1_A) \circ k = \langle g \circ (j_1 \circ k), j_2 \circ k \rangle$. Then, by$(**)$ we have

$$f \circ (p_1 \circ h) = g \circ (j_1 \circ k) \;\text{ and }\; p_2 \circ h = j_2 \circ k.$$

Now, consider the following diagram:

$$\begin{array}{ccccc} & & X & & \\ p_1 \circ h \swarrow & & \downarrow !u & \searrow j_1 \circ k & \\ B \underset{q_1}{\longleftarrow} & & P & \underset{q_2}{\longrightarrow} C & \\ f \searrow & & & \swarrow g & \\ & & D & & \end{array} \qquad \begin{array}{ccc} & X & \\ u \swarrow & \downarrow \langle u, \varkappa \rangle & \searrow p_2 \circ h = \varkappa \\ P \underset{\eta_1}{\longleftarrow} & P \times A & \underset{\eta_2}{\longrightarrow} A \end{array}$$

Since $f \circ (p_1 \circ h) = g \circ (j_1 \circ k)$, i.e. the outer diagram commutes, $\exists! u : X \longrightarrow P$ with $q_1 \circ u = p_1 \circ h$ and $q_2 \circ u = j_1 \circ k$. As it is clearly seen from the above product diagram, we also have $\eta_1 \circ \langle u, p_2 \circ h \rangle = u$ and $\eta_2 \circ \langle u, p_2 \circ h \rangle = p_2 \circ h$.

Let us have a look the below pullback diagram now.

$$\begin{array}{ccccc} & & X & & \\ & h \swarrow & \downarrow \langle u, \varkappa \rangle & \searrow k & \\ B \times A & \underleftarrow{q_1 \times 1_A} & P \times A & \underrightarrow{q_2 \times 1_A} & C \times A \\ & f \times 1_A \searrow & & \swarrow g \times 1_A & \\ & & D \times A & & \end{array}$$

Since

$$\begin{aligned} (q_1 \times 1_A) \circ (u, p_2 \circ h) &= \langle q_1 \circ u, p_2 \circ h \rangle \\ &= \langle p_1 \circ h, p_2 \circ h \rangle \\ &= h \end{aligned}$$

and also

$$
\begin{aligned}
(q_2 \times 1_A) \circ (u, p_2 \circ h) &= \langle q_2 \circ u, p_2 \circ h \rangle \\
&= \langle j_1 \circ k, j_2 \circ k \rangle \\
&= k
\end{aligned}
$$

$\langle u, p_2 \circ h \rangle$ makes the above diagram commute. Hence, existence part is satisfied.

As for uniqueness part, for $v : X \longrightarrow P \times A$ let $(q_1 \circ 1_A) \circ v = h$ and $(q_2 \times 1_A) \circ v = k$. Then, we have

$$
\begin{array}{ccccc}
 & & X & & \\
\eta_1 \circ v \swarrow & & \downarrow v & & \searrow \eta_2 \circ v \\
P & \underset{\eta_1}{\longleftarrow} & P \times A & \underset{\eta_2}{\longrightarrow} & A \\
q_1 \downarrow & & \downarrow q_1 \times 1_A & & \downarrow 1_A \\
B & \underset{p_1}{\longleftarrow} & B \times A & \underset{p_2}{\longrightarrow} & A
\end{array}
\qquad
\begin{array}{ccccc}
 & & X & & \\
\eta_1 \circ v \swarrow & & \downarrow v & & \searrow \eta_2 \circ v \\
P & \underset{\eta_1}{\longleftarrow} & P \times A & \underset{\eta_2}{\longrightarrow} & A \\
q_2 \downarrow & & \downarrow q_2 \times 1_A & & \downarrow 1_A \\
C & \underset{r_1}{\longleftarrow} & C \times A & \underset{r_2}{\longrightarrow} & A
\end{array}
$$

$\langle q \circ (\eta_1 \circ v), \eta_2 \circ v \rangle = (q_1 \times 1_A) \circ v = h$ and $\langle q_2 \circ (\eta_1 \circ v), \eta_2 \circ v \rangle = (q_2 \times 1_A) \circ v = k$.

On the other hand,

$$
\begin{array}{ccccc}
 & & X & & \\
p_1 \circ h \swarrow & & \downarrow h & & \searrow p_2 \circ h \\
B & \underset{p_1}{\longleftarrow} & B \times A & \underset{p_2}{\longrightarrow} & A
\end{array}
\qquad
\begin{array}{ccccc}
 & & X & & \\
j_1 \circ k \swarrow & & \downarrow k & & \searrow j_2 \circ k \\
C & \underset{j_1}{\longleftarrow} & C \times A & \underset{j_2}{\longrightarrow} & A
\end{array}
$$

$h = \langle p_1 \circ h, p_2 \circ h \rangle$ and $k = \langle j_1 \circ k, j_2 \circ k \rangle$. So, we have $q_1 \circ (\eta_1 \circ v) = p_1 \circ h$, $p_2 \circ h = \eta_2 \circ v = j_2 \circ k$ and $q_2 \circ (\eta_1 \circ v) = j_1 \circ k$.

$$
\begin{array}{ccccc}
 & & X & & \\
 & p_1 \circ h \swarrow & \downarrow !u & \searrow j_1 \circ k & \\
 & B \underset{q_1}{\longleftarrow} & P & \underset{q_2}{\longrightarrow} C & \\
 & f \searrow & & \swarrow g & \\
 & & D & &
\end{array}
$$

Since $u : X \longrightarrow P$ is the unique arrow with $q_1 \circ u = p_1 \circ h$ and $q_2 \circ u = j_1 \circ k$, $u = \eta_1 \circ v$. Hence, $\langle u, p_2 \circ h \rangle = \langle \eta_1 \circ v, \eta_2 \circ v \rangle = v$. As a result, universality condition also holds. So, $(P \times A, q_1 \times 1_A, q_2 \times 1_A) = pb(f \times 1_A, g \times 1_A)$.

As a result, the functor $(\_) \times A$,for any set A, preserves pullbacks and hence weak pullbacks as well.

3.3 Natural Transformations

The coalgebraic semantics connects with the algebraic semantics by natural transformations. Below we give the definition of natural transformations and an example for it.

Definition 3.3.1 Given a parallel pair of functors $F, G : \mathbf{A} \longrightarrow \mathbf{B}$, a *natural transformation* from F to G is a family of arrows $v_A : FA \longrightarrow GA$, one for each object A in $\mathbf{A}$, such that for every arrow $f : A \longrightarrow A'$ of $\mathbf{A}$ the following square commutes:

$$
\begin{array}{ccccc}
A & \quad & FA & \xrightarrow{v_A} & GA \\
\downarrow f & & Ff \downarrow & & \downarrow Gf \\
A' & & FA' & \xrightarrow{v_{A'}} & GA'
\end{array}
$$

We write $v : F \longrightarrow G$ for the natural transformation and call the arrows v_A the components of v. If the above diagram appears in a diagram we may say that it commutes by naturality of v.

Example 3.3.2 Let U : **Grp** $\longrightarrow$ **Set** be the forgetful functor and let S : **Grp** $\longrightarrow$ **Set** be the "squaring functor", defined by

$$\mathbf{Grp} \xrightarrow{\quad S \quad} \mathbf{Set}$$

$$\begin{array}{ll} \mathcal{G} & S(\mathcal{G}) = G \times G \ni (x, y) \\ \downarrow f & \downarrow S(f) = f \times f \\ \mathcal{H} & S(\mathcal{H}) = H \times H \ni (f(x), f(y)). \end{array}$$

For each group $\mathcal{G} = \langle G, . \rangle$, its operation is a function $\tau_G : G^2 \longrightarrow G$, denoted by for any $x, y \in G$ in $\mathcal{G}$, $\tau_G(x, y) = x.y \in G$ ($\bullet$). Now, we want to show that $\{\tau_G : G^2 \longrightarrow G\}_G$ is a natural transformation from S to U. First, consider the below diagram:

$$\mathbf{Grp} \quad \begin{array}{c} \xrightarrow{\quad S \quad} \\ \downarrow \tau \\ \xrightarrow{\quad U \quad} \end{array} \quad \mathbf{Set}$$

$$\begin{array}{llcl} \mathcal{G} & & G^2 = S(\mathcal{G}) \xrightarrow{\quad \tau_G \quad} & U(\mathcal{G}) = G \\ \downarrow f \quad S(f) & f \times f = & S(f) \downarrow & \downarrow U(f) = f \\ \mathcal{H} & & H^2 = S(\mathcal{G}) \xrightarrow[\quad \tau_H \quad]{} & U(\mathcal{G}) = H \end{array}$$

So, it is enough to show that the above diagram commutes, i.e. in equations for any $f : \mathcal{G} \longrightarrow \mathcal{H}$, $U(f) \circ \tau_G = \tau_H \circ S(f)$.

In **Grp**, for any $x, y \in G$ in $\mathcal{G}$ and for any group homomorphism $G \xrightarrow{f} H$ we have

$$f(x.y) = f(x).f(y) \quad (*)$$

where the first operation is performed in $\mathcal{G}$ and the second in $\mathcal{H}$ because homomorphisms preserve the structure. Hence, for any $x, y \in G$,

$$\begin{array}{lcl} (U(f) \circ \tau_G)(x,y) & = & (f \circ \tau_G)(x,y) \\ by\ (\bullet) & = & f(x.y) \\ by\ (*) & = & f(x).f(y) \\ by\ (\bullet) & = & \tau_H\ (f(x), f(y)) \\ & = & (\tau_H\ \circ (f \times f))(x,y) \\ & = & (\tau_H \circ S(f))(x,y). \end{array}$$

As a result, we obtain $U(f) \circ \tau_G = \tau_H \circ S(f)$ for any $G \xrightarrow{f} H$ which means that the above diagram commutes so, the family $\tau = \{\tau_G\}_{G\ in\ \mathbf{Grp}}$ is a natural transformation from S to U. The naturality condition simply means that $f(x.y) = f(x).f(y)$ for any group homomorphism $G \xrightarrow{f} H$ and any $x, y \in G$ in $\mathcal{G}$. Thus, "operation" in groups can be regarded as a natural transformation.

3.4 Definition of Coalgebras and Coalgebra Morphisms

Definition 3.4.1 Let $\mathbf{C}$ be a category and $T : \mathbf{C} \longrightarrow \mathbf{C}$ be a functor. Then, a *coalgebra* is a pair (X, γ) where X in $\mathbf{C}$ is the *carrier* and $\gamma : X \longrightarrow T(X)$ the *transition structure* of the coalgebra.

Throughout in this thesis we will consider coalgebras for endofunctors over **Set**.

Given a $T-$coalgebra (X, γ), we refer to X as the set of states and to γ as the *coalgebra map* or the *successor function.*

The definition of bisimilarity and some associated propositions will be given by means of rooted $T-$coalgebras. So, I find it useful to mention rooted $T-$coalgebras briefly.

Definition 3.4.2 A rooted $T-$coalgebra is a pair $(\mathbf{X},x)$ where $\mathbf{X} = (X, \gamma)$ is a $T-$coalgebra and $x \in X$ is an element of X, the so-called "root of $\mathbf{X}$".

We give some examples of structures which naturally arise as coalgebras.

Example 3.4.3

(i) Let L be a set (of *labels*) and $T : \mathbf{Set} \longrightarrow \mathbf{Set}$ be the functor $L \times (\_)$ which takes a set X to the cartesian product $L \times X$. Then, a pair $(C, \gamma_C : C \longrightarrow L \times C)$ gives us a $T-$coalgebra.

(ii) If we take $L = A$ in (i) and choose the transition map as $\gamma = \langle hd, tl \rangle : A^\omega \longrightarrow A \times A^\omega$ where A^ω is the set of all infinite words over an alphabet A and $hd : A^\omega \longrightarrow A$, $tl : A^\omega \longrightarrow A^\omega$ then for any $x \in A^\omega$, $\gamma(x) = \langle hd(x), tl(x) \rangle \in A \times A^\omega$. So, we obtain a $T-$coalgebra structure $(A^\omega, \gamma : A^\omega \longrightarrow A \times A^\omega)$ and it provides us an infinite stream on A.

Example 3.4.4 The prime example for coalgebras are *Kripke frames* or *transition systems* from modal logic perspective.

(i) Kripke frames (W, R) correspond to the $\mathcal{P}-$coalgebras $(W, R[\_] : W \longrightarrow \mathcal{P}(W))$ where $R[\_] : W \longrightarrow \mathcal{P}(W)$ denotes the function that maps a state $w \in W$ to the set $R[w] \subseteq W$ of $R-$successors of w. A $\mathcal{P}-$coalgebra (X, γ), on the other hand, corresponds to the Kripke frame (X, R_γ) where $R_\gamma \subseteq X \times X$ is defined by putting $\forall x, y \in X \;\; xR_\gamma y \;\; :\Leftrightarrow y \in \gamma(x)$.

(ii) Labelled transition systems (LTSs), or more simply transition systems are a simple kind of relational structure. A LTS is a pair $(W, \{R_a \mid a \in A\})$ where W is a non-empty set of states, A is a non-empty set (of labels) and for each $a \in A$, $R_a \subseteq W \times W$. Transition systems can be viewed as an abstract model of computation: the states are the possible states of a computer, labels stand for programs and $(u, v) \in R_a$

means that there is an execution of the program a that starts in state u and terminates in state v.

(iii) (W, R_a, R_b, R_c) is a deterministic labelled transition system where, as it is seen from the below figure,

$$\begin{array}{ccccc} & & w_3 & & \\ & & \uparrow b & \nwarrow c & \\ w_1 & \overset{a}{\longrightarrow} & w_2 & & \overset{\curvearrowleft a}{w_4} \end{array}$$

$W = \{w_1, w_2, w_3, w_4\}$, $A = \{a, b, c\}$, $R_a = \{(w_1, w_2), (w_4, w_4)\}$, $R_b = \{(w_2, w_2)\}$ and $R_c = \{(w_4, w_3)\}$. The system is deterministic because for each state where it is possible to make a transition, it is fixed which state that transition will take us to.

(iv) The simplest deterministic systems are coalgebras of the identity functor.

$$\begin{array}{l} S \\ \downarrow \alpha_S \\ I(S) = S \end{array} \qquad s \longrightarrow_S s' \;:\Longleftrightarrow\; \alpha_S(s) = s'.$$

Hence, any $I-$coalgebra (S, α_S) and the transition system $(S, \longrightarrow_S)$ coincide. $\alpha_S : S \longrightarrow I(S)$ gives the dynamics of the system and should be read as: in a state s, the system S can make a transition step to the state s'. Moreover, $\forall s \in S$, $\exists! s' \in S$ such that $s \longrightarrow_S s'$ because otherwise $\alpha_S : S \longrightarrow I(S)$ is not a function. Thus, such systems are deterministic.

(v) A non-deterministic LTS is as the following:

$$\begin{array}{ccccc} w_0 & & w_3 & & \\ \uparrow a & & \uparrow b & \nwarrow c & \\ w_1 & \overset{a}{\longrightarrow} & w_2 & \overset{a}{\longleftarrow} & \overset{\curvearrowleft a}{w_4} \end{array}$$

In this figure, a is a non-deterministic program for if we execute it in state w_4 we either loop back into w_4 or move to w_2, that is the state we will reach is not fixed. Similarly, if we execute it in w_1 we move to either w_0 or w_2.

(vi) Non-deterministic systems can be represented in many different ways. The simplest non-deterministic systems, in which several transitions may be possible from one state, are the coalgebras of the power set functor as it is shown in the following figure.

$$\begin{array}{l} S \\ \downarrow \alpha_S \qquad s \longrightarrow_S s^{'} \;\; :\Leftrightarrow \;\; s^{'} \in \alpha_S(s) \\ \mathcal{P}(S) \end{array}$$

In the following example we show that how coalgebras model image-finite systems.

Example 3.4.5

(i) Let $F(S) = (\mathcal{P}_\omega(S))^A$ where A is any set then consider the $F-$coalgebras $(S, \gamma : S \longrightarrow (\mathcal{P}_\omega(S))^A)$ where the transition structure γ on S is interpreted as follows:

$$\begin{array}{rccccl} \gamma : & S & \longrightarrow & (\mathcal{P}_\omega(S))^A & & \\ & s & \longmapsto & \gamma(s) : & A & \longrightarrow \mathcal{P}_\omega(S) \\ & & & & a & \longmapsto \left\{ s^{'} \in S \mid s \stackrel{a}{\longrightarrow} s^{'} \right\} \end{array}$$

where $|\gamma(s)(a)| < \omega$ for any $a \in A$. This condition may include the possibility of $\left|\gamma(s)(a^{'})\right| = 0$ for some $a^{'} \in A$. Thus, $F-$coalgebras can be used to model *image-finite* labelled transition systems while $\mathcal{P}_\omega-$coalgebras represent image-finite unlabelled ones. *Image-finiteness* property means: for every $s \in S$ and $a \in A$, the number of reachable states from s, denoted by $\left\{ s^{'} \mid s \stackrel{a}{\longrightarrow} s^{'} \right\}$, is finite but observe that we are not putting any restrictions on the total number of different labels $a \in A$, that is on the cardinality of A.

(ii) Image-finite labelled systems, that is given in (i), can also be given as the coalgebras of the functor $F(S) = \mathcal{P}_\omega(S^A)$. Consider any coalgebra $(S, \gamma : S \longrightarrow \mathcal{P}_\omega(S^A))$ where the transition map γ is defined as follows:

$$\begin{array}{rrcl} \gamma : & S & \longrightarrow & \mathcal{P}_\omega(S^A) \\ & s & \longmapsto & \gamma(s) = \{f_s \mid f_s : A \longrightarrow S\} \end{array}$$

So, for any $s \in S$, $\gamma(s) = \{f_s \mid f_s : A \longrightarrow S\}$ is finite. Hence, for each $a \in A$, the cardinality of $\{f_s(a) \mid f_s \in \gamma(s)\}$ for any $a \in A$, should be finite. Also, there is a very trivial case: for some $s \in S$, $\gamma(s)$ may equal to empty set and this means that s terminates, in other words, has no successor states. Therefore, both $\mathcal{P}_\omega((_)^A)-$coalgebras and $(\mathcal{P}_\omega(_))^A-$coalgebras model image-finite LTSs for the same set A.

Example 3.4.6 The contravariant power set functor can be used to model *hyper systems*, in which a state can make non-deterministically a step to the set of states:

$$\begin{array}{ll} S & \\ \downarrow \alpha_S & \qquad s \longrightarrow V \quad \text{iff} \quad V \in \alpha_S(s) \\ (\bar{P} \circ \bar{P})(S) & \end{array}$$

Thus, from any state s the system can reach $V \subseteq S$ but not necessarily each v in V. For example, there may be a state s from which we reach the empty set after a transition is performed if $\emptyset \in \alpha_S(s)$.

Definition 3.4.7 Let $\mathbf{C}$ be a category and $T : \mathbf{C} \longrightarrow \mathbf{C}$ be a functor. Then, a *$T-$coalgebra morphism* $f : (X, \gamma) \longrightarrow (Y, \delta)$ between two $T-$coalgebras (X, γ) and (Y, δ) is a morphism $f : X \longrightarrow Y$ in $\mathbf{C}$ such that the following diagram commutes that is, $\delta \circ f = Tf \circ \gamma$.

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & T(X) \\ \downarrow f & \circ & \downarrow Tf \\ Y & \xrightarrow[\delta]{} & T(Y) \end{array}$$

$T-$coalgebra morphisms are functions that preserve and reflect $T-$transition structures.

As we show in Example 3.4.8, the notion of a $T-$coalgebra morphism generalizes bounded morphisms of modal logic in a natural way.

Example 3.4.8 $\mathcal{P}$-coalgebra morphisms between any $\mathcal{P}$-coalgebras correspond to the bounded morphisms between corresponding Kripke frames. First of all, given any two Kripke frames (W_1, R_1), (W_2, R_2) and a bounded morphism $f : (W_1, R_1) \longrightarrow (W_2, R_2)$ between them, we obtain the corresponding $\mathcal{P}-$coalgebras and the $\mathcal{P}-$coalgebra morphism between them in the following way.

As we have shown in Example 3.4.4 (i), (W_1, R_1) and (W_2, R_2) correspond to the $\mathcal{P}-$coalgebras $(W_{1,}\ \gamma : W_1 \longrightarrow \mathcal{P}(W_1))$ and $(W_{2,}\ \delta : W_2 \longrightarrow \mathcal{P}(W_2))$ respectively where $\gamma = R_1[\_]$ and $\delta = R_2[\_]$. Now, we want to show that the bounded morphism f gives us a $\mathcal{P}-$coalgebra morphism between these $\mathcal{P}-$coalgebras.

$$\begin{array}{ccc} W_1 & \xrightarrow{\gamma} & \mathcal{P}(W_1) \\ f\downarrow & \circ & \downarrow \mathcal{P}f \\ W_2 & \xrightarrow[\delta]{} & \mathcal{P}(W_2) \end{array}$$

Hence, we need to show that the above diagram commutes, that is $\delta \circ f = \mathcal{P}f \circ \gamma$ where $\mathcal{P}f = f[\_]$ denotes the *direct image* function. So, it is enough to prove for any $w_1 \in W_1$, $(\delta \circ f)(w_1) = (\mathcal{P}f \circ \gamma)(w_1)$. Since for any w_2,

$$\begin{array}{lll} w_2 \in (\delta \circ f)(w_1) & \text{iff} & w_2 \in \delta(f(w_1)) \\ & \text{iff} & w_2 \in R_2[f(w_1)] \\ & \text{iff} & (f(w_1), w_2) \in R_2 \\ \text{(by } (ii)\text{)} & \text{iff} & \exists v \in W_1 \ni (w_1, v) \in R_1 \ and \ f(v) = w_2 \\ & \text{iff} & \exists v \in W_1 \ni v \in R_1[w] \ \ and \ f(v) = w_2 \\ & \text{iff} & \exists v \in W_1 \ni \ f(v) = w_2 \in f[\gamma(w_1)] \\ & \text{iff} & w_2 \in (\mathcal{P}f \circ \gamma)(w_1), \end{array}$$

the above diagram commutes which means that $f : W_1 \longrightarrow W_2$ is a $\mathcal{P}-$coalgebra morphism between (W_1, γ) and (W_2, δ).

Now, let's show the converse. Let (X, γ) and (Y, δ) be two $\mathcal{P}-$coalgebras and $f : X \longrightarrow Y$ be a $\mathcal{P}-$coalgebra morphism between (X, γ) and (Y, δ). Then, we can define the corresponding Kripke frames (X, R_γ) and (Y, R_δ) as it is shown before. Now, we want to show that $f : X \longrightarrow Y$ is a bounded morphism between (X, R_γ) and (Y, R_δ). Since f is a $\mathcal{P}-$coalgebra morphism, the following diagram commutes:

$$\begin{array}{rcl} X & \xrightarrow{\gamma} & \mathcal{P}(X) \\ f\downarrow & \circ & \downarrow \mathcal{P}f \\ Y & \xrightarrow[\delta]{} & \mathcal{P}(Y) \end{array}$$

In equations, $\delta \circ f = (\mathcal{P}f) \circ \gamma.(*)$

(i)For $x_1, x_2 \in X$, let $(x_1, x_2) \in R_\gamma$ then $x_2 \in \gamma(x_1)$ by definition. Then,

$$\begin{array}{lcl} \mathcal{P}f(\gamma(x_1)) & = & f[\gamma(x_1)] \\ \text{(by } (*)) & = & (\delta \circ f)(x_1) \\ & = & \delta(f(x_1)) \\ & = & R_\delta[f(x_1)]. \end{array}$$

Hence, $\exists z \in f[\gamma(x_1)] \ni f(x_2) = z$ and $(f(x_1), z) \in R_\delta$. As a result, forth (or homomorphic) condition holds.

(ii)For $x_1 \in X$, $z \in Y$, let $(f(x_1), z) \in R_\delta$. Then,

$$\begin{array}{lcl} R_\delta[f(x_1)] & = & \delta(f(x_1)) \\ \text{(by } (*)) & = & (\mathcal{P}f \circ \gamma)(x_1) \\ & = & \mathcal{P}f(\gamma(x_1)) \\ & = & f[\gamma(x_1)]. \end{array}$$

Hence, since $z \in R_\delta\left[f(x_1)\right]$, $\exists w \in \gamma(x_1) \ni f(w) = z$, that is $\exists w \in X$ with $(x_1, w) \in R_\gamma$ and $f(w) = z$. Consequently, back condition is also satisfied.

So, **Coalg**$(\mathcal{P})$ is the category which consists of Kripke frames as objects and bounded morphisms as arrows.

In the following example we give an equivalent condition for $F-$coalgebra morphisms.

Example 3.4.9 Let $(S, \longrightarrow_S, A)$ and $(T, \longrightarrow_T, A)$ be two labelled transition systems with the same set of labels A. Assume that $F : \mathbf{Set} \longrightarrow \mathbf{Set}$ is the functor $\mathcal{P}(A \times \_)$. Let (S, α_S) and (T, α_T) be the corresponding representations as $F-$systems. Then, by definition, an $F-$coalgebra morphism $f : (S, \alpha_S) \longrightarrow (T, \alpha_T)$ is a function $f : S \longrightarrow T$ with $F(f) \circ \alpha_S = \alpha_T \circ f$ where $F(f) = \mathcal{P}(A \times f)$. Now, we verify that $F(f) \circ \alpha_S = \alpha_T \circ f$ is equivalent to the following two conditions:

(i)For all s' in S, if $s \xrightarrow{a}_S s'$ then $f(s) \xrightarrow{a}_T f(s')$.

(ii)For all t in T, if $f(s) \xrightarrow{a}_T t$ then $\exists s'$ in S with $s \xrightarrow{a}_S s'$ and $f(s') = t$.

First of all, assume that $F(f) \circ \alpha_S = \alpha_T \circ f$. $(*)$

Proof of (i): Let $s' \in S$ be such that $s \xrightarrow{a}_S s'$ then $(a, s') \in \alpha_S(s)$. Thus, $(a, f(s')) \in (F(f) \circ \alpha_S)(s)$. So, by $(*)$, $(a, f(s')) \in \alpha_T(f(s))$ but then $f(s) \xrightarrow{a}_T f(s')$.

Proof of (ii): Let $t \in T$ be such that $f(s) \xrightarrow{a}_T t$ then $(a, t) \in \alpha_T(f(s))$. Again, from $(*)$ we obtain $(a, t) \in Ff(\alpha_S(s))$. Hence, $\exists s' \in S \ni f(s') = t$ and $(a, s') \in \alpha_S(s)$ but then $s \xrightarrow{a}_S s'$ and $f(s') = t$.

Conversely, suppose that the above conditions $(i) - (ii)$ hold. We want to show that $F(f) \circ \alpha_S = \alpha_T \circ f$.

Let $s \in S$ then it is enough to show that $(F(f) \circ \alpha_S)(s) = (\alpha_T \circ f)(s)$.

Case(1):

Assume that $s \in S$ is not $\longrightarrow_S$related. Then, $\alpha_S(s) = \emptyset$, so $F(f)(\emptyset) = (Ff \circ \alpha_S)(s) = \emptyset$. On the other hand, $f(s)$ is not $\longrightarrow_T$related either i.e. $\alpha_T(f(s)) = \emptyset$. Otherwise, if $\alpha_T(f(s)) \neq \emptyset$ then $\exists (a, t) \in \alpha_T(f(s))$ i.e. $f(s) \xrightarrow{a}_T t$ .So, by (ii) $\exists s' \in S \ni$

$f(s^{'}) = t$ and $s \xrightarrow{a}_S s^{'}$ which means $(a, s^{'}) \in \alpha_S(s)$. $(\longrightarrow\longleftarrow)$. As a result, $(F(f) \circ \alpha_S)(s) = (\alpha_T \circ f)(s)$.

Case(2):

Let $\alpha_S(s) \neq \emptyset$.

$(\subseteq)$: Let $(a, t) \in (F(f) \circ \alpha_S)(s)$ then $\exists s^{'} \in S \ni f(s^{'}) = t$ and $(a, s^{'}) \in \alpha_S(s)$. So, $s \xrightarrow{a}_S s^{'}$ but then by (i) $f(s) \xrightarrow{a}_T f(s^{'})$.Hence, $(a, f(s^{'})) = (a, t) \in \alpha_T(f(s))$.

So, $(F(f) \circ \alpha_S)(s) \subseteq (\alpha_T \circ f)(s)$.

$(\supseteq)$: Let $(a, t) \in \alpha_T(f(s))$ then $f(s) \xrightarrow{a}_T t$. Thus, by (ii) $\exists s^{'} \in S \ni s \xrightarrow{a}_S s^{'}$ where $f(s^{'}) = t$ i.e. for some $s^{'} \in S \ni f(s^{'}) = t$, $(a, s^{'}) \in \alpha_S(s)$. So, $(a, f(s^{'})) = (a, t) \in (F(f) \circ \alpha_S)(s)$.

Therefore, $(\alpha_T \circ f)(s) \subseteq (F(f) \circ \alpha_S)(s)$.

Thus, it is now obvious that an $F-$coalgebra morphism is a transition preserving and reflecting function.

3.5 Category of Coalgebras

We define the category of $F-$coalgebras and $F-$coalgebra morphisms for a functor $F : \mathbf{Set} \longrightarrow \mathbf{Set}$.

Given an $F-$coalgebra $(S, \alpha : S \longrightarrow F(S))$, the identity function 1_S on S is always an $F-$ coalgebra homomorphism because the below diagram commutes, that is $\alpha \circ 1_S = 1_{F(S)} \circ \alpha = F(1_S) \circ \alpha$.

$$\begin{array}{ccc} S & \xrightarrow{\alpha} & F(S) \\ 1_S \downarrow & \circ & \downarrow\ F(1_S) \\ S & \xrightarrow[\alpha]{} & F(S) \end{array}$$

Also, given $F-$coalgebras $(S, \alpha_S), (T, \alpha_T), (A, \alpha_A)$ and $F-$coalgebra morphisms $f : (S, \alpha_S) \longrightarrow (T, \alpha_T)$, $g : (T, \alpha_T) \longrightarrow (A, \alpha_A)$ $g \circ f : S \longrightarrow A$ is also an $F-$coalgebra morphism. Consider the below diagram:

$$\begin{array}{rcl} S & \xrightarrow{\alpha_S} & F(S) \\ f \downarrow & \circ & \downarrow Ff \\ T & \xrightarrow{\alpha_T} & F(T) \\ g \downarrow & \circ & \downarrow Fg \\ A & \xrightarrow{\alpha_A} & F(A) \end{array}$$

Since f and g are $F-$coalgebra morphisms, the above diagrams commute. So, the whole diagram commutes as well. Thus,

$$\begin{aligned} \alpha_A \circ (g \circ f) &= F(g) \circ (\alpha_T \circ f) \\ &= (F(g) \circ F(f)) \circ \alpha_S \\ &= F(g \circ f) \circ \alpha_S. \end{aligned}$$

Hence, $g \circ f : (S, \alpha_S) \longrightarrow (A, \alpha_A)$ is an $F-$coalgebra morphism. This composition is associative and given an $F-$ coalgebra morphism $f : (S, \alpha_S) \longrightarrow (T, \alpha_T)$, $f \circ 1_S = 1_T \circ f = f$.

As a result, collection of all $F-$coalgebras together with $F-$coalgebra morphisms is a category, denoted by $\mathbf{Set}_F$. It can also be represented with

$$\mathbf{Coalg}(F) := \langle \text{all } F - \text{coalgebras, all } F - \text{coalgebra morphisms} \rangle .$$

3.6 Bisimulations of Coalgebras

One way of looking at coalgebras is that a coalgebra consists of some set of states X and the coalgebra map $\gamma : X \longrightarrow TX$ allows us to observe certain properties of these states. Coalgebra morphisms preserve and reflect observable properties of objects i.e., for a coalgebra morphism $f : (X, \gamma) \longrightarrow (Y, \delta)$ we want x and $f(x)$ are observably equivalent. It turns out that this notion of behavioural equivalence generalizes existing notions of bisimilarity and hence we refer to it as $T-$bisimilarity.

Definition 3.6.1 Let $T : \mathbf{Set} \longrightarrow \mathbf{Set}$ be an endofunctor and $(X_1, \gamma_1), (X_2, \gamma_2)$ be $T-$coalgebras. We say that two states $x_1 \in X_1$ and $x_2 \in X_2$ are *behaviourally equivalent* or $T-$*bisimilar* if there is a $T-$coalgebra (Y, δ) and $T-$coalgebra morphisms $f_1 : (X_1, \gamma_1) \longrightarrow (Y, \delta)$ and $f_2 : (X_2, \gamma_2) \longrightarrow (Y, \delta)$ such that $f_1(x_1) = f_2(x_2)$. In this case, we write $(X_1, \gamma_1), x_1 \Leftrightarrow_T (X_2, \gamma_2), x_2$.

Coalgebraic $\mathcal{P}-$bisimulations are exactly the bisimulations from modal logic for the language without propositional variables but before showing this let's give the definition of $F-$bisimulations between two $F-$coalgebras.

Definition 3.6.2 Let (S, α_S) and (T, α_T) be two $F-$coalgebras. Then a subset $R \subseteq S \times T$ is an $F-$*bisimulation* between S and T if there exists an $F-$transition structure $\alpha_R : R \longrightarrow F(R)$ such that the projections $\pi_1 : R \longrightarrow S$ and $\pi_2 : R \longrightarrow T$ are $F-$coalgebra morphisms, that is the following diagram commutes.

$$\begin{array}{ccc}
S & \xrightarrow{\quad \alpha_S \quad} & F(S) \\
\uparrow \pi_1 & \circ & \uparrow F(\pi_1) \\
\mathbf{R} & \overset{\alpha_R}{- - - - - >} & \mathbf{F(R)} \\
\downarrow \pi_2 & \circ & \downarrow F(\pi_2) \\
T & \xrightarrow{\quad \alpha_T \quad} & F(T)
\end{array}$$

In other words, $\alpha_S \circ \pi_1 = F(\pi_1) \circ \alpha_R$ and $\alpha_T \circ \pi_2 = F(\pi_2) \circ \alpha_R$ for some $\alpha_R : R \longrightarrow F(R)$. Then, we say that (R, α_R) is called a *bisimulation* between S and T. Hence, two states $s \in S$ and $t \in T$ are called bisimilar if there is a bisimulation R with $(s, t) \in R$.

Some remarks concerning the definitions of behavioural equivalence and bisimilarity are in order:

Bisimilarity always implies behavioural equivalence. However, converse of this statement is not true in general. Let us show this by means of the following counter-example.

Example 3.6.3 Let T be the endofunctor on **Set**, given by $T(X) = \{(x, y, z) \in X^3 \mid card\,\{x, y$
for any set X and $(C, \gamma : C \longrightarrow T(C))$ be a $T-$coalgebra. Then, since T does not allow for any observations we intuitively regard that all states in C are behaviourally equivalent. Let $x, y \in C$ then we claim that x and y are behaviourally equivalent but not bisimilar. Behavioural equivalence is clear, so let us show why bisimilarity is not satisfied between these states.

Assume for a contradiction that x and y are bisimilar. Hence, there exists a $T-$coalgebra (E, ξ), turning projections π_1 and π_2 into $T-$coalgebra morphisms which means that following diagram commutes. Moreover, $(x, y) \in E$.

$$\begin{array}{ccc}
C & \xrightarrow{\qquad\gamma\qquad} & T(C) \\
\pi_1 \uparrow & \circ & \uparrow T(\pi_1) \\
E & \xrightarrow{\qquad\xi\qquad} & T(E) \\
\pi_2 \downarrow & \circ & \downarrow T(\pi_2) \\
C & \xrightarrow{\qquad\gamma\qquad} & T(C)
\end{array}$$

So, we have $\gamma \circ \pi_1 = T(\pi_1) \circ \xi$ and $\gamma \circ \pi_2 = T(\pi_2) \circ \xi$. Since $(x, y) \in E$, $\gamma \circ \pi_1(x, y) = \gamma(x) = (x_1, x_2, x_3) \in T(C)$ and $\gamma \circ \pi_2(x, y) = \gamma(y) = (y_1, y_2, y_3)$ for some $x_i \in C$ and $y_i \in C$ $\forall i = 1, 2, 3$.Hence, by commutativity, there must be $\xi(x, y) \in T(E)$ such that $T(\pi_1)(\xi(x, y)) = \gamma(x) = (x_1, x_2, x_3)$ and $T(\pi_1)(\xi(x, y)) = \gamma(y) = (y_1, y_2, y_3)$ whereas $(\gamma(x), \gamma(y)) \notin T(E)$ because $T(E) = \{\langle (u_1, v_1), (u_2, v_2), (u_{3,} v_3) \rangle \in E^3 \mid card\,\{(u_1, v_1), (u_2, v_2), (u_{3,}$
So, we obtain a contradiction. Therefore, x and y are not bisimilar.

The definition of an $F-$*bisimulation* can also be reformulated as follows.

Fact 3.6.4 Let $F : \mathbf{Set} \longrightarrow \mathbf{Set}$ be a functor. Assume that $\mathbf{X} = (X, \gamma), \mathbf{Y} = (Y, \delta) \in \text{coalg}(F)$ and $Z \subseteq X \times Y$.Then, Z is an $F-$bisimulation iff for all $(x, y) \in Z$, we have $(\gamma(x), \delta(y)) \in F(Z)$.

Proof: Let $F : \mathbf{Set} \longrightarrow \mathbf{Set}$ be a functor. Assume that for $\mathbf{X} = (X, \gamma), \mathbf{Y} = (Y, \delta) \in \text{coalg}(F)$, $Z \subseteq X \times Y$.

$(\Rightarrow):$ Suppose that Z is an $F-$bisimulation between $\mathbf{X}$ and $\mathbf{Y}$. So, by definition, there exists an $F-$coalgebra $(Z, \xi : Z \longrightarrow F(Z))$ which turns the following projections π_1 and π_2 into $F-$coalgebra morphisms. Thus, the below diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\qquad\gamma\qquad} & F(X) \\ \uparrow \pi_1 & \circ & \uparrow F(\pi_1) \\ \mathbf{Z} & - - - \overset{\xi}{-} - - > & \mathbf{F(Z)} \\ \downarrow \pi_2 & \circ & \downarrow F(\pi_2) \\ T & \xrightarrow{\qquad\delta\qquad} & F(T) \end{array}$$

In equations, we have $\gamma \circ \pi_1 = F(\pi_1) \circ \xi$ and $\delta \circ \pi_2 = F(\pi_2) \circ \xi$. $(*)$ Let $(x, y) \in Z$ then we have by $(*)$

$$\begin{array}{lllllll} F(\pi_1)(\xi(x,y)) & = & (F(\pi_1) \circ \xi)(x,y) & = & (\gamma \circ \pi_1)(x,y) & = & \gamma(x) \in F(X) \quad \text{and} \\ F(\pi_2)(\xi(x,y)) & = & (F(\pi_2) \circ \xi)(x,y) & = & (\delta \circ \pi_2)(x,y) & = & \delta(y) \in F(Y). \end{array}$$

Thus, we obtain

$$\begin{array}{rcl} (\gamma(x), \delta(y)) & = & (F\pi_1(\xi(x,y)), F\pi_2(\xi(x,y))) \in FX \times FY \\ & = & (F\pi_1, F\pi_2)(\xi(x,y)) \in FX \times FY \\ & = & \xi(x,y). \end{array}$$

Since $\xi(x,y) \in F(Z)$ and $(\gamma(x), \delta(y)) = \xi(x,y)$, $(\gamma(x), \delta(y)) \in F(Z)$.

$(\Leftarrow):$ Now, assume that for all $(x, y) \in Z$, $\exists (\gamma(x), \delta(y)) \in F(Z)$. We claim that Z is an $F-$bisimulation between $\mathbf{X}$ and $\mathbf{Y}$.

By assumption, we are allowed to define the following map

$$\begin{array}{rlcl} \xi: & Z & \longrightarrow & F(Z) \\ & (x,y) & \longmapsto & \xi(x,y) = (\gamma(x), \delta(y)) \end{array}$$

which behaves as γ on $x \in X$ and δ on $y \in Y$. It is now enough to show that the below diagram commutes.

$$\begin{array}{lcl} X & \xrightarrow{\quad\quad\gamma\quad\quad} & F(X) \\ \uparrow \pi_1 & \circ & \uparrow F(\pi_1) \\ \mathbf{Z} & - - - \overset{\xi}{-} - - > & \mathbf{F}(\mathbf{Z}) \\ \downarrow \pi_2 & \circ & \downarrow F(\pi_2) \\ T & \xrightarrow{\quad\quad\delta\quad\quad} & F(T) \end{array}$$

Let $(x,y) \in Z$ then $(\gamma \circ \pi_1)(x,y) = \gamma(x) = F\pi_1(\gamma(x), \delta(y)) = (F\pi_1 \circ \xi)(x,y)$ and $(\delta \circ \pi_2)(x,y) = \delta(y) = F\pi_2(\gamma(x), \delta(y)) = (F\pi_2 \circ \xi)(x,y)$. As a result, $\gamma \circ \pi_1 = F\pi_1 \circ \xi$ and $\delta \circ \pi_2 = F\pi_2 \circ \xi$ which means that π_1 and π_2 are $F-$coalgebra morphisms.

As a result, $(Z, \xi) \in$ coalg(F) is an $F-$bisimulation between **X** and **Y**.

Let T : **Set** $\longrightarrow$ **Set** be a functor. If T preserves weak pullbacks then $T-$bisimulations match with the notion of $T-$bisimilarity in the following sense.

Fact 3.6.5 Let T : **Set** $\longrightarrow$ **Set** be a functor that preserves weak pullbacks and (**X**,x) $:= (X, \gamma, x)$, (**Y**,y) $:= (Y, \delta, y)$ be rooted $T-$coalgebras. Then, (**X**,x) and (**Y**,y) are $T-$bisimilar iff there is a $T-$bisimulation $Z \subseteq X \times Y$ between **X** and **Y** with $(x,y) \in Z$.

Proof Let T : **Set** $\longrightarrow$ **Set** preserve weak pullbacks and (**X**,x), (**Y**,y) be rooted $T-$coalgebras.

$(\Longrightarrow)$: Assume that (**X**,x)$\underline{\leftrightarrow}_T$(**Y**,$y$). Then, by definition, there is a rooted $T-$coalgebra (**E**, e) where **E** $=(E, \xi)$ and also there exists $T-$coalgebra morphisms f : **X** $\longrightarrow$ **E** and g : **Y** $\longrightarrow$ **E** with $f(x) = g(y) = e$. So, the below diagram commutes.

$$
\begin{array}{lcl}
X & \xrightarrow{\qquad\gamma\qquad} & T(X) \\
\downarrow f & \circ & \downarrow T(f) \\
\mathbf{E} & - - - - \overset{\xi}{-} - - - > & T(\mathbf{E}) \\
\uparrow g & \circ & \uparrow T(g) \\
Y & \xrightarrow{\qquad\delta\qquad} & T(Y)
\end{array}
$$

On the other hand, we know that in **Set** any corner of arrows $A \xrightarrow{f} C \xleftarrow{g} B$ always has a pullback P with the projection arrows π_1 and π_2 which is denoted by $P := \{(a, b) \in A \times B \mid f(a) = g(b)\} \subseteq A \times B$. So, there is also a pullback of functions f and g, say (Z, π_1, π_2) where $Z = \{(x, y) \in X \times Y \mid f(x) = g(y)\} \subseteq X \times Y$. Hence, as $f(x) = g(y) = e$, $(x, y) \in Z$.

Since any pullback is also a weak pullback, (Z, π_1, π_2) is a weak pullback of f and g as well but T preserves weak pullbacks. Thus, $(T(Z), T(\pi_1), T(\pi_2)) = wpb(T(f), T(g))$.

$$
\mathbf{Set} \xrightarrow{\qquad\qquad T \qquad\qquad} \mathbf{Set}
$$

$$
\begin{array}{lcl}
Z & \xrightarrow{\pi_2} & Y \\
\pi_1 \downarrow & & \downarrow g \\
X & \xrightarrow[f]{} & E
\end{array}
\qquad\qquad\qquad
\begin{array}{lcl}
T(Z) & \xrightarrow{T(\pi_2)} & T(Y) \\
T(\pi_1) \downarrow & & \downarrow T(g) \\
T(X) & \xrightarrow[T(f)]{} & T(E)
\end{array}
$$

Now, consider the below diagram:

$$
\begin{array}{rcl}
Z & \xrightarrow{\qquad\zeta\qquad} & T(Z) \\
\pi_2 \downarrow & & \downarrow T(\pi_2) \\
Y & \xrightarrow{\qquad\delta\qquad} & T(Y) \\
g \downarrow & & \downarrow T(g) \\
E & \xrightarrow{\qquad\xi\qquad} & T(E) \\
f \uparrow & & \uparrow T(f) \\
X & \xrightarrow{\qquad\gamma\qquad} & T(X) \\
\pi_1 \uparrow & & \uparrow T(\pi_1) \\
Z & \xrightarrow{\qquad\zeta\qquad} & T(Z)
\end{array}
$$

For $\gamma \circ \pi_1 : Z \longrightarrow T(X)$ and $\delta \circ \pi_2 : Z \longrightarrow T(Y)$,

$$\begin{aligned}
T(f) \circ (\gamma \circ \pi_1) &= (T(f) \circ \gamma) \circ \pi_1 \\
&= \xi \circ (f \circ \pi_1) \qquad \text{(since } f \text{ is a } T-\text{coalgebra morphism)} \\
&= (\xi \circ g) \circ \pi_2 \qquad \text{(since } (Z, \pi_1, \pi_2) = pb(f, g)) \\
&= T(g) \circ (\delta \circ \pi_2). \quad \text{(since } g : \mathbf{Y} \longrightarrow \mathbf{E} \in \text{coalg}(T))
\end{aligned}$$

Thus, since $(T(Z), T(\pi_1), T(\pi_2)) = wpb(T(f), T(g))$ $\exists \zeta : Z \longrightarrow T(Z)$ with $T(\pi_1) \circ \zeta = \gamma \circ \pi_1$ and $T(\pi_2) \circ \zeta = \delta \circ \pi_2$ which turns π_1 and π_2 into $T-$coalgebra morphisms. So, $(Z, \zeta : Z \longrightarrow T(Z))$ is a bisimulation between $\mathbf{X}$ and $\mathbf{Y}$ and $(x, y) \in Z$.

Example 3.6.6 Let T be the powerset functor $\mathcal{P}$.Then, $\mathcal{P}-$bisimulations coincide with the standard notion of bisimulation for transition systems, that is given two $\mathcal{P}-$coalgebras (X, γ) and (Y, δ), a relation $Z \subseteq X \times Y$ is a $\mathcal{P}-$bisimulation between (X, γ) and (Y, δ) iff $(x, y) \in Z$ implies

(i) for all $x^{'} \in \gamma(x)$ there is a $y^{'} \in \delta(y)$ with $(x^{'}, y^{'}) \in Z$ and

(ii) for all $y^{'} \in \delta(y)$ there is an $x^{'} \in \gamma(x)$ with $(x^{'}, y^{'}) \in Z$.

Now, let's verify this. Let (X, γ) and (Y, δ) be two $\mathcal{P}-$coalgebras.

$(\Longrightarrow)$: Assume that a relation $Z \subseteq X \times Y$ is a $\mathcal{P}-$bisimulation between (X, γ) and (Y, δ). Hence, there is a transition structure ξ on Z which turns π_x and π_y into $\mathcal{P}-$coalgebra morphisms. Then, we have $\gamma \circ \pi_x = \mathcal{P}(\pi_x) \circ \xi$ and $\delta \circ \pi_y = \mathcal{P}(\pi_y) \circ \xi$. $(*)$

$$\begin{array}{rccl}
X & \xrightarrow{\quad\gamma\quad} & \mathcal{P}(X) \\
\pi_x \uparrow & \circ & \uparrow F(\pi_1) \\
Z & - - - \overset{\xi}{-} - - > & \mathcal{P}(Z) \\
\pi_y \downarrow & \circ & \downarrow F(\pi_2) \\
Y & \xrightarrow{\quad\delta\quad} & \mathcal{P}(Y)
\end{array}$$

Let $(x, y) \in Z$.

(i) Let $x^{'} \in \gamma(x)$ then by standard transition notation, $x \longrightarrow_X x^{'}$. Also, by $(*)$,

$$\begin{aligned} \gamma(x) &= (\gamma \circ \pi_x)(x, y) \\ &= (\mathcal{P}(\pi_x) \circ \xi)(x, y) \\ &= \pi_x \left[\xi(x, y)\right]. \end{aligned}$$

Since $x^{'} \in \gamma(x)$, $x^{'} \in \pi_x \left[\xi(x, y)\right]$. Again, by $(*)$,

$$\begin{aligned} \delta(y) &= (\delta \circ \pi_y)(x, y) \\ &= (\mathcal{P}(\pi_y) \circ \xi)(x, y) \\ &= \pi_y \left[\xi(x, y)\right]. \end{aligned}$$

Thus, since $x^{'} \in \pi_x \left[\xi(x, y)\right]$, $\delta(y) = \pi_y \left[\xi(x, y)\right]$ and $\xi(x, y) \subseteq Z$, $\exists y^{'} \in \delta(y) = \pi_y \left[\xi(x, y)\right] \ni (x^{'}, y^{'}) \in \xi(x, y)$. So, $(x^{'}, y^{'}) \in Z$. As a result, $\exists y^{'} \in Y \ni y \longrightarrow_Y y^{'}$ and $(x^{'}, y^{'}) \in Z$. This is nothing, but the forth condition which is satisfied.

(ii) Let $y^{'} \in \delta(y)$. Then, this can shown as $y \longrightarrow_Y y^{'}$ in another notation. With the same argument as in (i), we have $\gamma(x) = \pi_x \left[\xi(x, y)\right]$ and $\delta(y) = \pi_y \left[\xi(x, y)\right]$. Since $y^{'} \in \delta(y) = \pi_Y \left[\xi(x, y)\right]$, $\gamma(x) = \pi_x \left[\xi(x, y)\right]$ and $\xi(x, y) \subseteq Z$, $\exists x^{'} \in X \ni (x^{'}, y^{'}) \in \xi(x, y)$ and $x^{'} \in \gamma(x)$. So, for some $x^{'} \in \gamma(x)$, $(x^{'}, y^{'}) \in Z$, that is $\exists x^{'} \in X \ni x \longrightarrow_X x^{'}$ and $(x^{'}, y^{'}) \in Z$. This is the back condition which holds for transition systems.

$(\Longleftarrow)$: Conversely, suppose that (i) and (ii) hold. Now, let's define a map $\xi : Z \longrightarrow \mathcal{P}(Z)$ as follows: for $(x, y) \in Z$, let $\xi(x, y)$ contain all the below pairs of $X \times Y$. By (i), because $\forall x^{'} \in \gamma(x)$ $\exists y^{'} \in \delta(y)$ with $(x^{'}, y^{'}) \in Z$ put all such pairs into $\xi(x, y)$ and moreover $\forall y^{''} \in \delta(y)$, by (ii), $\exists x^{''} \in \gamma(x)$ with $(x^{''}, y^{''}) \in Z$. Next, put all these pairs into $\xi(x, y)$ as well. Now, we claim that $\xi : Z \longrightarrow \mathcal{P}(Z)$ turns π_x and π_y into $\mathcal{P}-$coalgebra morphisms i.e. $\gamma \circ \pi_x = \mathcal{P}(\pi_x) \circ \xi$ and $\delta \circ \pi_y = \mathcal{P}(\pi_y) \circ \xi$. Let $(x, y) \in Z$ then we need to show that $\gamma(x) = (\pi_x \circ \gamma)(x, y) = (\mathcal{P}(\pi_x) \circ \xi)(x, y) = \pi_x \left[\xi(x, y)\right]$ (1) and $\delta(y) = (\pi_y \circ \delta)(x, y) = (\mathcal{P}(\pi_y) \circ \xi)(x, y) = \pi_y \left[\xi(x, y)\right]$. (2) For any $x^{'}$,

$$\begin{aligned} x' \in \gamma(x) &\text{ iff } \exists y' \in \delta(y) \ni (x', y') \in \xi(x,y) \\ &\text{ iff } x' \in \pi_x\left[\xi(x,y)\right]. \end{aligned}$$

For any y'',

$$\begin{aligned} y'' \in \delta(y) &\text{ iff } \exists x'' \in \gamma(x) \ni (x'', y'') \in \xi(x,y) \\ &\text{ iff } y'' \in \pi_y\left[\xi(x,y)\right]. \end{aligned}$$

Both of these are satisfied by the construction of $\xi : Z \longrightarrow \mathcal{P}(Z)$. As a result, $\exists (Z,\xi) \in \text{coalg}(\mathcal{P})$ which turns π_x and π_y into $\mathcal{P}-$coalgebra morphisms which implies that $Z \subseteq X \times Y$ with the transition structure ξ on Z behaves as a bismulation between (X,γ) and (Y,δ).

Now, given any two $\mathcal{P}-$coalgebras (X,γ) and (Y,δ), let's consider corresponding Kripke frames (X,R_γ) and (Y,R_δ). Assume that there is a $\mathcal{P}-$bisimulation $Z \subseteq X \times Y$ betwen (X,γ) and (Y,δ). Then, the above conditions (i) and (ii) hold but they can also be interpreted as follows:

For any $(x,y) \in Z$,

(i) for all $x' \in R_\gamma[x]$, $\exists y' \in R_\delta[y] \ni (x',y') \in Z$ i.e. for any $x' \in X$, $(x,x') \in R_\gamma \Longrightarrow \exists y' \in Y \ni (y,y') \in R_\delta$ and $(x',y') \in Z$ but this is the usual notation of forth condition from modal logic.

(ii) for all $y' \in R_\delta[y]$, $\exists x' \in R_\gamma(x) \ni (x',y') \in Z$ i.e. for any $y' \in Y$, $(y,y') \in R_\delta \Longrightarrow \exists x' \in X \ni (x,x') \in R_\gamma$ and $(x',y') \in Z$. Similarly, this condition is nothing, but the back condition from modal logic.

As a result, $Z \subseteq X \times Y$ is a bisimulation between Kripke frames (X,R_γ) and (Y,R_δ).

Conversely, given any two Kripke frames (X,R) and (Y,R') and a bisimulation $Z \subseteq X \times Y$ between these frames, consider corresponding $\mathcal{P}-$coalgebras (X,γ_R) and $(Y,\delta_{R'})$. Let $(x,y) \in Z$ then by forth condition, $\forall x' \in X$, $(x,x') \in R \Longrightarrow \exists y' \in Y \ni (y,y') \in R'$ and $(x',y') \in Z$ i.e. $\forall x' \in R[x] = \gamma_R[x]$, $\exists y' \in R'[y] = \delta_{R'}(y) \ni (x',y') \in Z$ which corresponds to (i).Now, by back condition, $\forall y'' \in Y$,

$(y, y'') \in R' \Longrightarrow \exists x'' \in X \ni (x, x'') \in R$ and $(x'', y'') \in Z$ i.e. $\forall y'' \in R'[y] = \delta_{R'}(y)$, $\exists x'' \in R[x] = \gamma_R(x) \ni (x'', y'') \in Z$ that corresponds to (ii) above. So, $Z \subseteq X \times Y$ is a $\mathcal{P}-$bisimulation between (X, γ_R) and $(Y, \delta_{R'})$. As a result, $\mathcal{P}-$bisimulations between any two $\mathcal{P}-$coalgebras correspond to the bisimulations between corresponding Kripke frames from modal logic.A similar example is given in Ex.3.6.7 for LTSs.

Example 3.6.7 Let $F(X) = \mathcal{P}(A \times X)$ for some set of labels. Now, consider any two labelled transition systems $(S, \longrightarrow_S, A)$ and $(T, \longrightarrow_T, A)$. Then, we claim that a $F-$bisimulation between the corresponding $F-$systems (S, α_S) and (T, α_T) is a relation $R \subseteq S \times T$ satisfying for all $\langle s, t\rangle \in R$,

(1) for all s' in S, if $s \xrightarrow{a}_S s'$ then there is t' in T with $t \xrightarrow{a}_T t'$ and $\langle s', t'\rangle \in R$.

(2) for all t in T, if $t \xrightarrow{a}_T t'$then there is s' in S with $s \xrightarrow{a}_S s'$ and $\langle s', t'\rangle \in R$.

The proof is so similar to the one in the above example so I will give it roughly.

Let R be a $F-$bisimulation with the transition structure $\alpha_R : R \longrightarrow F(R)$. As before, α_R induces a relation $\longrightarrow_R \subseteq R \times A \times R$. Let $\langle s, t\rangle \in R$.

Suppose that $s \xrightarrow{a}_S s'$ then $\pi_1 \langle s, t\rangle \xrightarrow{a}_S s'$ but π_1 is a homomorphism so $\exists \langle s'', t'\rangle \in R \ni \langle s, t\rangle \xrightarrow{a}_R \langle s'', t'\rangle$ and $\pi_1 \langle s'', t'\rangle = s'$. Thus, $\langle s', t'\rangle \in R$, Because π_2 is a homomorphism, we have $t \xrightarrow{a}_T t'$ which concludes the proof of (1). Similarly, assume that $t \xrightarrow{a}_T t'$ then $\pi_2 \langle s, t\rangle \xrightarrow{a}_T t'$ but $\pi_2 \in \mathrm{coalg}(F)$ so $\exists \langle s', t''\rangle \in R \ni \langle s, t\rangle \xrightarrow{a}_R \langle s', t''\rangle$ where $\pi_2 \langle s', t''\rangle = t'$. Since π_1 is a homomorphism, $s \xrightarrow{a}_S s'$ which completes the proof of (2).

Conversely, let (1) and (2) both hold for a relation $R \subseteq S \times T$. Then, define

$$\begin{array}{rlll} \alpha_R : & R & \longrightarrow & F(R) = \mathcal{P}(A \times R) \\ & \langle s, t\rangle & \longmapsto & \alpha_R(\langle s, t\rangle) = \left\{ (s', t') \in R \mid s \xrightarrow{a}_S s' \text{ and } t \xrightarrow{a}_T t' \right\}. \end{array}$$

Now, we need to show that (R, α_R) is a $F-$bisimulation between (S, α_S) and (T, α_T).

$$
\begin{array}{ccc}
S & \xrightarrow{\qquad \alpha_S \qquad} & F(S) \\
\uparrow \pi_1 & \circ & \uparrow F(\pi_1) \\
R & - - - - \overset{\alpha_R}{-} - - > & F(R) \\
\downarrow \pi_2 & \circ & \downarrow F(\pi_2) \\
T & \xrightarrow{\qquad \alpha_T \qquad} & F(T)
\end{array}
$$

Let $\langle s, t\rangle \in R$ then it is enough to show that $(\alpha_S \circ \pi_1)(s,t) = (F(\pi_1) \circ \alpha_R)(s,t)$ (i) and $(\alpha_T \circ \pi_2)(s,t) = (F(\pi_2) \circ \alpha_R)(s,t)$ (ii).

$(\alpha_S \circ \pi_1)(s,t) = \alpha_S(\pi_1(s,t)) = \alpha_S(s) \in F(S) = \mathcal{P}(A \times S)$. So, for any $(a, s^{'})$,

$$
\begin{array}{lll}
(a, s^{'}) \in (\alpha_S \circ \pi_1)(s,t) & \text{iff} & (a, s^{'}) \in \alpha_S(s) \\
 & \text{iff} & s \xrightarrow{a}_S s^{'} \\
\text{(by (1))} & \text{iff} & \exists t^{'} \in T \ni t \xrightarrow{a}_T t^{'} \text{ and } (s^{'}, t^{'}) \in R \\
 & \text{iff} & \exists t^{'} \in T \ni (s,t) \xrightarrow{a}_R (s^{'}, t^{'}) \\
 & \text{iff} & \exists t^{'} \in T \ni (a, (s^{'}, t^{'})) \in \alpha_R(s,t) \\
 & \text{iff} & (a, s^{'}) \in (F(\pi_1) \circ \alpha_R)(s,t).
\end{array}
$$

So, (i) is satisfied.

$(\alpha_T \circ \pi_2)(s,t) = \alpha_T(\pi_2(s,t)) = \alpha_T(t) \in F(T) = \mathcal{P}(A \times T)$. So, for any $(a, t^{'})$,

$$
\begin{array}{lll}
(a, t^{'}) \in (\alpha_T \circ \pi_2)(s,t) & \text{iff} & t \xrightarrow{a}_T t^{'} \\
\text{(by (2))} & \text{iff} & \exists s^{'} \in S \ni s \xrightarrow{a}_S s^{'} \text{ and } (s^{'}, t^{'}) \in R \\
 & \text{iff} & \exists s^{'} \in S \ni (s,t) \xrightarrow{a}_R (s^{'}, t^{'}) \\
 & \text{iff} & \exists s^{'} \in S \ni (a, (s^{'}, t^{'})) \in \alpha_R(s,t) \\
 & \text{iff} & (a, t^{'}) \in (F(\pi_2) \circ \alpha_R)(s,t).
\end{array}
$$

Hence, (ii) also holds. As a result, (R, α_R) is an $F-$bisimulation between (S, α_S) and (T, α_T).

Note that , in general, α_R is not the only transition structure on R which turns π_1 and π_2 into $F-$coalgebra morphisms i.e. it is not unique in general.

Chapter 4

Predicate Liftings

In this chapter we give the concept of predicate liftings and show how predicate liftings give rise to modal languages, interpreted over coalgebras.

4.1 Relationship Between Predicate Liftings and Coalgebras

In this section first we give the definition of a predicate lifting and some basic examples about it. Then we show the connections between Kripke frames and coalgebras via predicate liftings.

Definition 4.1.1 A *predicate lifting* λ for an endofunctor T on **Set** is an order preserving natural transformation $\lambda : \bar{\mathcal{P}} \longrightarrow \bar{\mathcal{P}} \circ T$ where $\bar{\mathcal{P}}$ is the contravariant power set functor. Spelling out this definition, a predicate lifting for T is an indexed family of maps $\lambda_C : \bar{\mathcal{P}}(C) \longrightarrow \bar{\mathcal{P}}(T(C))$ such that

$$
\begin{array}{cccccc}
 & \xrightarrow{\bar{\mathcal{P}}} & & & \\
\mathbf{Set} & \Downarrow \lambda & \mathbf{Set} & & \\
 & \xrightarrow[\bar{\mathcal{P}}\circ T]{} & & & \\
C & & \bar{\mathcal{P}}(C) & \xrightarrow{\lambda_C} & \bar{\mathcal{P}}(T(C)) \\
\downarrow f & & \bar{\mathcal{P}}(f)\uparrow & \circ & \uparrow \bar{\mathcal{P}}(T(f)) \\
D & & \bar{\mathcal{P}}(D) & \xrightarrow[\lambda_D]{} & \bar{\mathcal{P}}(T(D))
\end{array}
$$

for any $f : C \longrightarrow D$, the above diagram commutes, i.e., $\lambda_C \circ \bar{\mathcal{P}}(f) = \bar{\mathcal{P}}(T(f)) \circ \lambda_D$.

Question: Why are predicate liftings named as liftings?

First, let us remember the definition of a lifting in *Category Theory.* For any arrows f, g and h, h is a lifting of f to g if $g \circ h = f$.

Now, let us consider a predicate lifting $\lambda : \bar{\mathcal{P}} \longrightarrow \bar{\mathcal{P}} \circ T$:

$$
\begin{array}{cccccc}
 & \xrightarrow{\bar{\mathcal{P}}} & & & \\
\mathbf{Set} & \Downarrow \lambda & \mathbf{Set} & & \\
 & \xrightarrow[\bar{\mathcal{P}}\circ T]{} & & & \\
C & & \bar{\mathcal{P}}(C) & \xrightarrow{\lambda_C} & \bar{\mathcal{P}}(T(C)) \\
\downarrow f & & \bar{\mathcal{P}}(f)\uparrow & \circ & \uparrow \bar{\mathcal{P}}(T(f)) \\
D & & \bar{\mathcal{P}}(D) & \xrightarrow[\lambda_D]{} & \bar{\mathcal{P}}(T(D))
\end{array}
$$

Then, for any $f : C \longrightarrow D$ the above diagram commutes i.e. $\lambda_C \circ \bar{\mathcal{P}}(f) = \bar{\mathcal{P}}(T(f)) \circ \lambda_D$. Hence, in a categorical context λ_D behaves as a lifting of

$\lambda_C \circ \bar{\mathcal{P}}(f)$ to $\bar{\mathcal{P}}(T(f))$. $\lambda_D : \bar{\mathcal{P}}(Y) \longrightarrow \bar{\mathcal{P}}(T(Y))$ takes predicates (or subsets) of Y to predicates (or subsets) of $T(Y)$.

In a logical context, predicate liftings allow us to reason about the states of a system after a transition has been performed. Order preservation thus lets us to infer formulas involving successor states. This corresponds to the congruence rule

$$\varphi \vdash \psi \Longrightarrow \Box\varphi \vdash \Box\psi$$

of modal logic. In other words,

$$[|\varphi|] \subseteq [|\psi|] \Longrightarrow [|\Box\varphi|] \subseteq [|\Box\psi|]\,.$$

Since predicate liftings generalize the interpretation of $\Box -$ *operator* it is not difficult to see how order preservation (for any set C and $A, B \subseteq C$, $A \subseteq B \Longrightarrow \lambda_C(A) \subseteq \lambda_C(B)$ where $\{\lambda_C\}_{C \text{ a set}}$ is a predicate lifting for any set endofunctor T) gives us the above rule. Now, let us see some examples of predicate liftings and observe how they interpret modal operators and also propositional variables through these examples.

Example 4.1.2 Let $T(X) = L \times X$ where L is a set *(of labels)*. We have seen that any $T-$coalgebra $(C, \gamma : C \longrightarrow L \times C)$ can be used to model a deterministic labelled transition system . Now, let $(X, \alpha : X \longrightarrow L \times X)$ be a $T-$coalgebra in which α is defined by $\alpha = \langle hd, tl\rangle$ where $hd : X \longrightarrow L$ and $tl : X \longrightarrow X$.

$$\begin{array}{ccl} X & \xrightarrow{\alpha} & T(X) = L \times X \\ x_1 & \longmapsto & \alpha(x_1) = \langle hd(x_1), tl(x_1)\rangle\,. \end{array}$$

If we consider $\{hd(x_1), hd \circ tl(x_1), hd \circ tl \circ tl(x_1), ...\}$ for any $x_1 \in X$ then (X, α) gives us a set of stream (or sequence) on L.

Now, define an operation $\lambda : \bar{\mathcal{P}} \longrightarrow \bar{\mathcal{P}} \circ T$ putting by for any X,

$$\lambda_X : \bar{\mathcal{P}}(X) \longrightarrow \bar{\mathcal{P}}(T(X)) = \bar{\mathcal{P}}(L \times X)$$

where for any $U \subseteq X$, $\lambda_X(U) = \{\omega \in TX \mid \pi_2(\omega) \in U\}$.

This operation is a predicate lifting for T. Let us verify it. Let $f : X \longrightarrow Y$.

$$\mathbf{Set} \quad \overset{\bar{\mathcal{P}}}{\longrightarrow} \; \Downarrow \lambda \; \underset{\bar{\mathcal{P}}\circ T}{\longrightarrow} \quad \mathbf{Set}$$

$$\begin{array}{lccc}
C & \bar{\mathcal{P}}(C) & \xrightarrow{\lambda_C} & \bar{\mathcal{P}}(T(C)) \\
\downarrow f & \bar{\mathcal{P}}(f) \uparrow & \circ & \uparrow \bar{\mathcal{P}}(T(f)) \\
D & \bar{\mathcal{P}}(D) & \underset{\lambda_D}{\longrightarrow} & \bar{\mathcal{P}}(T(D))
\end{array}$$

First, we need to show $\lambda_X \circ \bar{\mathcal{P}}(f) = \bar{\mathcal{P}}(T(f)) \circ \lambda_Y$, i.e., the naturality of the above diagram. Since, for any $V \subseteq Y$,

$$\begin{array}{rcl}
(\bar{\mathcal{P}}(T(f)) \circ \lambda_D)(V) & = & (Tf)^{-1}[\lambda_D(V)] \\
 & = & \{\omega \in TX \mid (Tf)(\omega) \in \lambda_D(V)\} \\
\text{(by construction of } \lambda) & = & \{\omega \in L \times X \mid \pi_2((Tf)(\omega)) \in V\} \\
 & = & \{\omega \in L \times X \mid (\pi_2 \circ (1_L \times f))(\omega) \in V\} \\
 & = & \{\omega \in L \times X \mid f(\pi_2(\omega)) \in V\} \\
\text{(by a Galois connection)} & = & \{\omega \in T(X) \mid \pi_2(\omega) \in f^{-1}[V]\} \\
\text{(by construction of } \lambda) & = & \lambda_X((f^{-1}[V]) \\
 & = & \lambda_X(\bar{\mathcal{P}}(f)(V)) \\
 & = & (\lambda_X \circ \bar{\mathcal{P}}(f))(V),
\end{array}$$

naturality is satisfied.

Recall: For any sets U, V and a function $f : U \longrightarrow V$, we frequently encounter direct $f[\_]$ and inverse $f^{-1}[\_]$ images in this thesis. They are related by a *Galois connection*: for $A \subseteq U$ and $B \subseteq V$, we have

$$f[A] \subseteq B \text{ iff } A \subseteq f^{-1}[B].$$

As for order preservation of λ, let $A, B \subseteq X$ be such that $A \subseteq B$ then we have by construction of λ,

$$\begin{aligned}
\lambda_X(A) &= \{\omega \in T(X) \mid \pi_2(w) \in A\} \\
&\subseteq \{\omega \in T(X) \mid \pi_2(w) \in B\} \\
&= \lambda_X(B)
\end{aligned}$$

Hence, order preservation holds as well. Consequently, the operation $\lambda : \bar{\mathcal{P}} \longrightarrow \bar{\mathcal{P}} \circ T$ is a predicate lifting for T.

Now, let us observe how such maps $\lambda_X : \bar{\mathcal{P}}(X) \longrightarrow \bar{\mathcal{P}}(T(X))$, for any set X, interpret the modal operator $\Box$.

Let $[|\varphi|] \subseteq X$, for any formula φ and then consider $\alpha^{-1} \circ \lambda_X : \bar{\mathcal{P}}(X) \longrightarrow \bar{\mathcal{P}}(X)$. So,

$$\begin{aligned}
(\alpha^{-1} \circ \lambda_X)([|\varphi|]) &= \{x \in X \mid \alpha(x) \in \lambda_X([|\varphi|])\} \\
\text{(by construction of } \lambda) \quad &= \{x \in X \mid \pi_2(\alpha(x)) \in [|\varphi|]\} \\
&= \{x \in X \mid tl(x) \in [|\varphi|]\} \\
&= \{x \in X \mid R[x] \subseteq [|\varphi|]\} \\
\text{(by definition of } \ell_R) \quad &= \ell_R([|\varphi|]) \\
\text{(by definition of } \Box - \text{operator)} \quad &= [|\Box\varphi|]
\end{aligned}$$

where $R[\_]$ denotes the set of all successor states of a point and for each set X, $\ell_R : \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ is defined by putting for any $A \subseteq \dot{X}$,

$$\ell_R(A) = \{x \in X \mid R[x] \subseteq A\}.$$

Hence, $(\alpha^{-1} \circ \lambda_X)([|\varphi|]) = [|\Box\varphi|]$.

Example 4.1.3 Let T be the power set functor $\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set}$.

Consider a $T-$coalgebra $(X, \delta : X \longrightarrow \mathcal{P}(X))$. Since for any $x \in X$, $\delta(x) \subseteq X$ we define a binary relation $R_\delta \subseteq X \times X$ as follows:

$$(x, y) \in R_\delta \quad :\Longleftrightarrow \quad y \in \delta(x)$$

Hence, we obtain the corresponding *Kripke frame* (X, R_δ) and so we can view the $T-$coalgebra (X, δ) as a *Kripke frame* (X, R_δ) .

Conversely, let $\mathcal{F} = (W, R)$ be a *Kripke frame* then since $R \subseteq X \times X$, we define a map $\alpha_R : W \longrightarrow \mathcal{P}(W)$ by putting for any $w \in W$,

$$\alpha_R(w) = \{w^{'} \in W \mid (w, w^{'}) \in R\}.$$

Hence, we obtain the corresponding $\mathcal{P}-$coalgebra $(W, \alpha_R : W \longrightarrow \mathcal{P}(W))$. Therefore, we can conceive the *Kripke frame* $\mathcal{F} = (W, R)$ as a $\mathcal{P}-$ coalgebra (W, α_R). This shows that, there is a one-to-one correspondence between $\mathcal{P}-$coalgebras and Kripke frames.

Now, define an operation $\lambda : \bar{\mathcal{P}} \longrightarrow \bar{\mathcal{P}} \circ \mathcal{P}$ putting by for any set X,

$$\begin{array}{lll} \bar{\mathcal{P}}(X) & \xrightarrow{\lambda_X} & \bar{\mathcal{P}}(\mathcal{P}(X)) \\ U & \longmapsto & \lambda_X(U) = \{V \in \mathcal{P}(X) \mid V \subseteq U\} = \mathcal{P}(U). \end{array}$$

We show that $\{\lambda_X\}_{X\ a\ set}$ is a predicate lifting for $\mathcal{P}$.

$$
\begin{array}{ccc}
 & \xrightarrow{\bar{\mathcal{P}}} & \\
\mathbf{Set} & \Downarrow \lambda & \mathbf{Set} \\
 & \xrightarrow[\bar{\mathcal{P}} \circ \mathcal{P}]{} &
\end{array}
$$

$$
\begin{array}{lcccc}
X & & \bar{\mathcal{P}}(X) & \xrightarrow{\lambda_X} & \bar{\mathcal{P}}(\mathcal{P}(X)) \\
\downarrow f & & \bar{\mathcal{P}}(f) \uparrow & \circ & \uparrow \bar{\mathcal{P}}(\mathcal{P}(f)) \\
Y & & \bar{\mathcal{P}}(Y) & \xrightarrow[\lambda_Y]{} & \bar{\mathcal{P}}(\mathcal{P}(Y))
\end{array}
$$

For $A, B \subseteq X$, let $A \subseteq B$ then by construction of λ,

$$
\begin{array}{rcl}
\lambda_X(A) & = & \mathcal{P}(A) \\
 & \subseteq & \mathcal{P}(B) \\
 & = & \lambda_X(B).
\end{array}
$$

Hence, order preservation holds.

Now, we need to show that for $f : X \longrightarrow Y$ and $V \subseteq Y$, $(\bar{\mathcal{P}}(\mathcal{P}(f)) \circ \lambda_Y)(V) = (\lambda_X \circ \bar{\mathcal{P}}(f))(V)$.

$$
\begin{array}{rcl}
(\bar{\mathcal{P}}(\mathcal{P}(f)) \circ \lambda_Y)(V) & = & (\mathcal{P}f)^{-1}\left[\lambda_Y(V)\right] \\
 & = & \{W \in \mathcal{P}(X) \mid (\mathcal{P}f)(W) \in \lambda_Y(V)\} \\
\text{(by construction of } \lambda) & = & \{W \in \mathcal{P}(X) \mid f\left[W\right] \in \mathcal{P}(V)\} \\
 & = & \{W \in \mathcal{P}(X) \mid f\left[W\right] \subseteq V\} \\
\text{(by a Galois connection)} & = & \{W \subseteq X \mid W \subseteq f^{-1}\left[V\right]\} \\
 & = & \mathcal{P}(f^{-1}\left[V\right]) \\
\text{(by construction of } \lambda) & = & \lambda_X(f^{-1}\left[V\right]) \\
 & = & (\lambda_X \circ \bar{\mathcal{P}}(f))(V).
\end{array}
$$

Therefore, $\{\lambda_X\}_{X \ a \ set}$ is a predicate lifting for $\mathcal{P}$.

$$
\begin{aligned}
\lambda_X(A) &= \mathcal{P}(A) \\
&\subseteq \mathcal{P}(B) \\
&= \lambda_X(B).
\end{aligned}
$$

Hence, order preservation holds.

Now, we need to show that for $f : X \longrightarrow Y$ and $V \subseteq Y$, $(\bar{\mathcal{P}}(\mathcal{P}(f)) \circ \lambda_Y)(V) = (\lambda_X \circ \bar{\mathcal{P}}(f))(V)$.

$$
\begin{array}{rcl}
(\bar{\mathcal{P}}(\mathcal{P}(f)) \circ \lambda_Y)(V) & = & (\mathcal{P}f)^{-1}[\lambda_Y(V)] \\
 & = & \{W \in \mathcal{P}(X) \mid (\mathcal{P}f)(W) \in \lambda_Y(V)\} \\
\text{(by construction of } \lambda) & = & \{W \in \mathcal{P}(X) \mid f[W] \in \mathcal{P}(V)\} \\
 & = & \{W \in \mathcal{P}(X) \mid f[W] \subseteq V\} \\
\text{(by a Galois connection)} & = & \{W \subseteq X \mid W \subseteq f^{-1}[V]\} \\
 & = & \mathcal{P}(f^{-1}[V]) \\
\text{(by construction of } \lambda) & = & \lambda_X(f^{-1}[V]) \\
 & = & (\lambda_X \circ \bar{\mathcal{P}}(f))(V).
\end{array}
$$

Therefore, $\{\lambda_X\}_{X \text{ a set}}$ is a predicate lifting for $\mathcal{P}$.

Example 4.1.4 Let $T(X) = \mathcal{P}(X) \times \mathcal{P}(\Phi)$ where Φ is a set (of *propositional variables* of a given modal logic). Consider an operation $\lambda_{(\_)} : \bar{\mathcal{P}}(\_) \longrightarrow \bar{\mathcal{P}}(\mathcal{P}(\_) \times \mathcal{P}(\Phi))$. So, for any set C, let

$$
\begin{array}{lcl}
\bar{\mathcal{P}}(C) & \xrightarrow{\lambda_C} & \bar{\mathcal{P}}(\mathcal{P}(C) \times \mathcal{P}(\Phi)) \\
X & \longmapsto & \lambda_C(X) = \{(A, B) \in T(C) \mid A \subseteq X\} = \mathcal{P}(X) \times \mathcal{P}(\Phi).
\end{array}
$$

$\{\lambda_C\}_{C \text{ a set}}$ is a predicate lifting for T. Let us verify it.

First, let $f : C \longrightarrow D$ and $X_1, X_2 \subseteq C$ be such that $X_1 \subseteq X_2$. Then, by construction of λ,

$$
\begin{array}{rcl}
\lambda_C(X_1) & = & \mathcal{P}(X_1) \times \mathcal{P}(\Phi) \\
 & \subseteq & \mathcal{P}(X_2) \times \mathcal{P}(\Phi) \\
 & = & \lambda_C(X_2).
\end{array}
$$

Thus, order preservation is satisfied.

Next, we have to show the naturality of λ. It is sufficient to show that the below diagram commutes:

$$
\begin{array}{cccccc}
 & \xrightarrow{\bar{\mathcal{P}}} & & & \\
\mathbf{Set} & \Downarrow \lambda & \mathbf{Set} & & \\
 & \underset{\bar{\mathcal{P}} \circ T}{\longrightarrow} & & & \\
C & & \bar{\mathcal{P}}(C) & \xrightarrow{\lambda_C} & \bar{\mathcal{P}}(T(C)) \\
\downarrow f & & \bar{\mathcal{P}}(f) \uparrow & \circ & \uparrow \bar{\mathcal{P}}(T(f)) \\
D & & \bar{\mathcal{P}}(D) & \underset{\lambda_D}{\longrightarrow} & \bar{\mathcal{P}}(T(D))
\end{array}
$$

For any $V \in \bar{\mathcal{P}}(D)$,

$$
\begin{array}{rcl}
(\bar{\mathcal{P}}(T(f)) \circ \lambda_D)(V) & = & (Tf)^{-1}\left[\lambda_D(V)\right] \\
 & = & \{\omega \in TC \mid (Tf)(\omega) \in \lambda_D(V)\} \\
\text{(by construction of } \lambda) & = & \{(A, B) \in TC \mid (Tf)(A, B) \in \mathcal{P}(V) \times \mathcal{P}(\Phi)\} \\
 & = & \left\{(A, B) \in TC \mid (\mathcal{P}(f) \times 1_{\mathcal{P}(\Phi)})(A, B) \in \mathcal{P}(V) \times \mathcal{P}(\Phi)\right\} \\
 & = & \{(A, B) \in TC \mid (f\left[A\right], B) \in \mathcal{P}(V) \times \mathcal{P}(\Phi)\} \\
 & = & \{(A, B) \in TC \mid f\left[A\right] \subseteq V\} \\
\text{(by a Galois connection)} & = & \{(A, B) \in TC \mid A \subseteq f^{-1}\left[V\right]\} \\
 & = & \mathcal{P}(f^{-1}\left[V\right]) \times \mathcal{P}(\Phi) \\
\text{(by construction of } \lambda) & = & \lambda_C(f^{-1}\left[V\right]) \\
 & = & (\lambda_C \circ \bar{\mathcal{P}}(f))(V).
\end{array}
$$

Hence, $(\bar{\mathcal{P}}(T(f)) \circ \lambda_D) = (\lambda_C \circ \bar{\mathcal{P}}(f))$, i.e., λ is a natural map.

Therefore, $\left\{\lambda_C : \bar{\mathcal{P}}(C) \longrightarrow \bar{\mathcal{P}}(\mathcal{P}(C) \times \mathcal{P}(\Phi))\right\}_C$ is a predicate lifting for T.

Now, consider a $T-$coalgebra $(X, \gamma : X \longrightarrow \mathcal{P}(X) \times \mathcal{P}(\Phi))$. We have seen before that there is a one-to-one correspondence between $T-$coalgebras and Kripke models. So, let $M_\gamma = (X, R_\gamma, V_\gamma)$ be the corresponding Kripke model and $[|\varphi|] \subseteq X$ be the semantics of a modal formula φ of a given modal language. We claim that $(\gamma^{-1} \circ \lambda_X)([|\varphi|]) = \{x \in X \mid \pi_1(\gamma(x)) \subseteq [|\varphi|] \;\; \textit{and} \;\; \pi_2(\gamma(x) \subseteq \Phi\}$ (where $\pi_1 : \mathcal{P}(X) \times \mathcal{P}(\Phi) \longrightarrow \mathcal{P}(X)$ denotes the first projection and $\pi_2 : \mathcal{P}(X) \times \mathcal{P}(\Phi) \longrightarrow \mathcal{P}(X)$ the second one) corresponds to the interpretation of the semantics of the modal formula $\Box\varphi$. Because

$$
\begin{array}{lcl}
(\gamma^{-1} \circ \lambda_X)([|\varphi|]) & = & \{x \in X \mid \gamma(x) \in \lambda_X([|\varphi|])\} \\
\text{(by construction of } \lambda) & = & \{x \in X \mid \gamma(x) \in \mathcal{P}([|\varphi|]) \times \mathcal{P}(\Phi)\} \\
 & = & \{x \in X \mid \pi_1(\gamma(x)) \subseteq [|\varphi|] \;\textit{and}\; \pi_2(\gamma(x)) \subseteq \Phi\} \\
 & = & \{x \in X \mid R_\gamma[x] \subseteq [|\varphi|]\} \\
 & = & \{x \in X \mid \forall y(R_\gamma xy \longrightarrow M_\gamma, y \Vdash \varphi)\} \\
\text{(by definition of } \ell_{R_\gamma}) & = & \ell_{R_\gamma}([|\varphi|]) \\
\text{(by definition of } \Box - \text{operator)} & = & [|\Box\varphi|]\,,
\end{array}
$$

the predicate lifting $\left\{\lambda_C : \bar{\mathcal{P}}(C) \longrightarrow \bar{\mathcal{P}}(\mathcal{P}(C) \times \mathcal{P}(\Phi))\right\}_C$ generalizes the box-operator.

The definition of $\lambda_C : \bar{\mathcal{P}}(C) \longrightarrow \bar{\mathcal{P}}(\mathcal{P}(C) \times \mathcal{P}(\Phi))$, for any set C, could also be rewritten in the following way: for any $U \subseteq C$,

$$
\begin{aligned}
\lambda_C(U) &= \{(A, B) \in \mathcal{P}(C) \times \mathcal{P}(\Phi) \mid A \subseteq U\} \\
&= \{(A, B) \in \mathcal{P}(C) \times \mathcal{P}(\Phi) \mid \pi_1(A, B) \subseteq U\} \\
&= \{\omega \in T(C) \mid \pi_1(\omega) \subseteq U\}
\end{aligned}
$$

We, now, claim that the family of maps $\{\pi_{1C} : \mathcal{P}(C) \times \mathcal{P}(\Phi) \longrightarrow \mathcal{P}(C)\}_C$, each defined as a first projection, is a natural transformation.

$$\mathbf{Set} \quad \begin{array}{c} \mathcal{P}(\_) \times \mathcal{P}(\Phi) \\ \xrightarrow{\hspace{4em}} \\ \Downarrow \pi_{1(\_)} \\ \xrightarrow{\hspace{5em}} \\ \mathcal{P}(\_) \end{array} \quad \mathbf{Set}$$

$$\begin{array}{ccccc} C & \qquad & \mathcal{P}(C) \times \mathcal{P}(\Phi) & \xrightarrow{\pi_{1C}} & \mathcal{P}(C) \\ \downarrow f & & \mathcal{P}(f) \times 1_{\mathcal{P}(\Phi)} \downarrow & \circ & \downarrow \mathcal{P}(f) \\ D & & \mathcal{P}(D) \times \mathcal{P}(\Phi) & \underset{\pi_{1D}}{\longrightarrow} & \mathcal{P}(D) \end{array}$$

Let $f : C \longrightarrow D$ and $(U, P) \in \mathcal{P}(C) \times \mathcal{P}(\Phi)$ then because

$$\begin{array}{ll} (\mathcal{P}(f) \circ \pi_{1C})(U, P) & = \mathcal{P}(f)(U) \\ & = f\,[U] \\ \text{(by definition of } \pi_1) & = \pi_{1D}\ (f\,[U]\,, P) \\ & = (\pi_{1D}\ \circ \mathcal{P}(f) \times 1_{P(\Phi)})(U, P), \end{array}$$

we have $\mathcal{P}(f) \circ \pi_{1C} = \pi_{1D} \circ (\mathcal{P}(f) \times 1_{P(\Phi)})$.

So, the above diagram commutes, i.e., the naturality of π_1 holds.

Thus, since we have defined λ by means of π_1, the naturality of π_1 implies the naturality of λ. Then, replacing π_1 by an arbitrary natural transformation, a *construction principle* for predicate liftings is obtained (see $[P]$ Prop3.3).

Proposition 4.1.5 Suppose $\mu : T \longrightarrow \mathcal{P}$ is a natural transformation where $\mathcal{P} :$ **Set** $\longrightarrow$ **Set** is the power set functor. Then, the operation $\lambda_C : \bar{\mathcal{P}}(C) \longrightarrow \bar{\mathcal{P}}(T(C))$ for any set C, given by $\lambda_C(U) = \{\omega \in T(C) \mid \mu_C(\omega) \subseteq U\}$, defines a predicate lifting for T.

Proof: Let $\mu : T \longrightarrow \mathcal{P}$ be a natural transformation. Then, define an operation $\lambda_{(\_)} : \bar{\mathcal{P}}(\_) \longrightarrow \bar{\mathcal{P}}(T(\_))$ as follows: for any set C,

$$\begin{array}{lcl}\bar{\mathcal{P}}(C) & \xrightarrow{\lambda_C} & \bar{\mathcal{P}}(T(C)) \\ U & \longmapsto & \lambda_C(U) = \{\omega \in T(C) \mid \mu_C(\omega) \subseteq U\}.\end{array}$$

Now, we need to show that $\left\{\lambda_C : \bar{\mathcal{P}}(C) \longrightarrow \bar{\mathcal{P}}(T(C))\right\}_C$ constitutes a predicate lifting for T.

Let $X_1, X_2 \subseteq C$ be such that $X_1 \subseteq X_2$. Since by construction of λ,

$$\begin{array}{rcl}\lambda_C(X_1) & = & \{\omega \in T(C) \mid \mu_C(\omega) \subseteq X_1\} \\ & \subseteq & \{\omega \in T(C) \mid \mu_C(\omega) \subseteq X_2\} \\ & = & \lambda_C(X_2),\end{array}$$

the order preservation holds for λ.

Let $f : C \longrightarrow D$ also be given.

$$\mathbf{Set} \underset{\bar{\mathcal{P}} \circ T}{\overset{\bar{\mathcal{P}}}{\Downarrow \lambda}} \mathbf{Set}$$

$$\begin{array}{ccccc} C & & \bar{\mathcal{P}}(C) & \xrightarrow{\lambda_C} & \bar{\mathcal{P}}(T(C)) \\ \downarrow f & & \bar{\mathcal{P}}(f) \uparrow & \circ & \uparrow \bar{\mathcal{P}}(T(f)) \\ D & & \bar{\mathcal{P}}(D) & \xrightarrow[\lambda_D]{} & \bar{\mathcal{P}}(T(D)) \end{array}$$

We claim that the above diagram commutes, i.e., $\bar{\mathcal{P}}(T(f)) \circ \lambda_D = \lambda_C \circ \bar{\mathcal{P}}(f)$.

Because μ is a natural transformation, the below diagram commutes. So, we have $\mathcal{P}(f) \circ \mu_C = \mu_D \circ T(f) \quad (*)$.

$$
\begin{array}{ccc}
 & \xrightarrow{T} & \\
\mathbf{Set} & \Downarrow \mu & \mathbf{Set} \\
 & \xrightarrow[\mathcal{P}]{} &
\end{array}
$$

$$
\begin{array}{ccccc}
C & & T(C) & \xrightarrow{\mu_C} & \mathcal{P}(C) \\
\downarrow f & & T(f)\downarrow & \circ & \downarrow \mathcal{P}(f) \\
D & & T(D) & \xrightarrow[\mu_D]{} & \mathcal{P}(D)
\end{array}
$$

Then, since for any $V \in \bar{\mathcal{P}}(D)$,

$$
\begin{array}{lcl}
(\bar{\mathcal{P}}(T(f)) \circ \lambda_D)(V) & = & (T(f))^{-1}\left[\lambda_D(V)\right] \\
 & = & \{\omega \in T(C) \mid (Tf)(\omega) \in \lambda_D(V)\} \\
\text{(by construction of } \lambda) & = & \{\omega \in T(C) \mid \mu_D((Tf)(\omega)) \subseteq V\} \\
\text{(by } (*), \text{ i.e.,the naturality of } \mu) & = & \{\omega \in T(C) \mid P(f)(\mu_C(\omega)) \subseteq V\} \\
 & = & \{\omega \in T(C) \mid f\left[\mu_C(\omega)\right] \subseteq V\} \\
\text{(by a Galois connection)} & = & \{\omega \in T(C) \mid \mu_C(\omega) \subseteq f^{-1}\left[V\right]\} \\
\text{(by construction of } \lambda) & = & \lambda_C(f^{-1}\left[V\right]) \\
 & = & (\lambda_C \circ \bar{\mathcal{P}}(f))\,(V)\,,
\end{array}
$$

the naturality of λ is also satisfied. So, $\left\{\lambda_C : \bar{\mathcal{P}}(C) \longrightarrow \bar{\mathcal{P}}((T(C))\right\}_{C\ \ a\ set}$ is a predicate lifting for T.

This proposition shows that for each natural transformation $\mu : T \longrightarrow \mathcal{P}$ it is possible to construct a corresponding predicate lifting for T. Since modal operators are interpreted by predicate liftings this proposition gives us a principle of obtaining new $\Box-$operators. In other words, for a given endofunctor T on **Set** such natural transformations $\mu : T \longrightarrow \mathcal{P}$ help us to find new predicate liftings for this endofunctor. So, new logics are found as a result of adding new modal operators to the language which means that variety of the same system increases.

In the above proposition, the naturality of the corresponding predicate liftings is ensured by the naturality of such natural transformations $\mu : T \longrightarrow \mathcal{P}$. So, we can interpret modal operators using these natural transformations in a slightly different way. Now, let us see how this procedure works through the below argument.

First, we consider the following diagram and then observe how we can use these natural transformations $\mu : T \longrightarrow \mathcal{P}$ in place of the corresponding predicate liftings in giving the semantics of a box-formula.

$$\begin{array}{ccccc}
\mathbf{Set} & \overset{T(_)}{\underset{\mathcal{P}(_)}{\overset{\Downarrow\, \mu_{(_)}}{\longrightarrow\!\!\!\!\longrightarrow}}} & \mathbf{Set} & & \\
C & \overset{\gamma}{\longrightarrow} & T(C) & \overset{\mu_C}{\longrightarrow} & \mathcal{P}(C) \\
\cup\,| & & \cup\,| & & \cup\,| \\
(\gamma)^{-1}\left[(\mu_C)^{-1}\left[\mathcal{P}([|\varphi|])\right]\ \right] & & (\mu_C)^{-1}\left[\mathcal{P}([|\varphi|])\right] & & \mathcal{P}([|\varphi|])
\end{array}$$

Let $(C, \gamma : C \longrightarrow T(C))$ be a $T-$coalgebra. Then, we have $(*)$

$$\begin{array}{rcl}
(\mu_C)^{-1}\left[\mathcal{P}([|\varphi|])\right] & = & \{\omega \in T(C) \mid \mu_C(\omega) \in \mathcal{P}([|\varphi|]\} \\
& = & \{\omega \in T(C) \mid \mu_C(\omega) \subseteq [|\varphi|]\} \\
\text{(by construction of } \lambda) & = & \lambda_C([|\varphi|])
\end{array}$$

If we continue this argumentation, because we know $[|\Box\varphi|] = ((\gamma)^{-1} \circ \lambda_C)([|\varphi|])$ in (C, γ) we obtain another representation of the semantics of a box-formula by means of a natural transformation $\mu : T \longrightarrow \mathcal{P}$ in the following way:

$$\begin{array}{rcll}
[|\Box\varphi|] & = & ((\gamma)^{-1} \circ \lambda_C)([|\varphi|]) & \\
\text{(by } (*)) & = & ((\gamma)^{-1} \circ (\mu_C)^{-1})\left[\mathcal{P}([|\varphi|])\right] & (= (\mu_C \circ \gamma)^{-1}\left[\mathcal{P}([|\varphi|])\right]) \\
& = & (\bar{\mathcal{P}}(\gamma) \circ \bar{\mathcal{P}}(\mu_C))(\mathcal{P}([|\varphi|])) & \\
(\bar{\mathcal{P}} \text{ preserves compositions.}) & = & (\bar{\mathcal{P}}(\mu_C \circ \gamma))(\mathcal{P}([|\varphi|])) &
\end{array}$$

So, we also have $[|\Box\varphi|] = (\bar{\mathcal{P}}(\mu_C \circ \gamma))(\mathcal{P}([|\varphi|]))$.

We show how predicate liftings can be used to interpret atomic propositions of Kripke models in the following example (see $[P]$ $Ex3.4.$).

Example 4.1.6 Let, for any set C, $T(C) = \mathcal{P}(C) \times \mathcal{P}(\Phi)$ where Φ is a set (of atomic propositions of a given modal language). Then, for a fixed $p \in \Phi$, consider a constant operation:

$$\begin{array}{rcl} \bar{\mathcal{P}}(C) & \stackrel{\lambda^p_C}{\longrightarrow} & \bar{\mathcal{P}}(T(C)) \\ U & \longmapsto & \lambda^p_C(U) = \{(X, A) \in T(C) \mid p \in A\}\,. \quad (*) \end{array}$$

Now, we claim that $\{\lambda^p_C\}_C$ is a predicate lifting for T.

First, order preservation of λ^p holds because for any $X_1, X_2 \subseteq C$, $X_1 \subseteq X_2$ implies

$$\begin{array}{lrcl} \text{by } (*) & \lambda^p_C(X_1) & = & \{(X, A) \in T(C) \mid p \in A\} \\ \text{by } (*) & & = & \lambda^p_C(X_2). \end{array}$$

Hence, we have $\lambda^p_C(X_1) \subseteq \lambda^p_C(X_2)$.

Then, we need to show the naturality of λ^p.

$$\mathbf{Set} \;\; \begin{array}{c} \stackrel{\bar{\mathcal{P}}}{\longrightarrow} \\ \Downarrow \lambda^p \\ \underset{\bar{\mathcal{P}}\circ T}{\longrightarrow} \end{array} \;\; \mathbf{Set}$$

$$\begin{array}{ccccc} C & \quad & \bar{\mathcal{P}}(C) & \stackrel{\lambda^p_C}{\longrightarrow} & \bar{\mathcal{P}}(T(C)) \\ \downarrow f & & \bar{\mathcal{P}}(f) \uparrow & \circ & \uparrow \bar{\mathcal{P}}(T(f)) \\ D & & \bar{\mathcal{P}}(D) & \underset{\lambda^p_D}{\longrightarrow} & \bar{\mathcal{P}}(T(D)) \end{array}$$

Let $f : C \longrightarrow D$ and $V \subseteq D$ be given. Since

$$\begin{array}{rcl}
(\bar{\mathcal{P}}(T(f)) \circ \lambda^p_D)(V) & = & (Tf)^{-1}\left[\lambda^p_D(V)\right] \\
& = & \{\omega \in T(C) \mid T(f)(\omega) \in \lambda^p_D(V)\} \\
& = & \left\{(X,A) \in TC \mid (\mathcal{P}(f) \times 1_{\mathcal{P}(\Phi)})(X,A) \in \lambda^p_D(V)\right\} \\
& = & \{(X,A) \in TC \mid (f\left[X\right], A) \in \lambda^p_D(V)\} \\
\text{(by construction of } \lambda^p\text{)} & = & \{(X,A) \in TC \mid p \in A\} \\
\text{(since } f^{-1}\left[V\right] \subseteq C \text{ and } \lambda^p_C \text{ is constant)} & = & \lambda^p_C(f^{-1}\left[V\right]) \\
& = & (\lambda^p_C \circ \bar{\mathcal{P}}(f))(V),
\end{array}$$

the above diagram commutes. Therefore, naturality of λ^p is also satisfied.

So, this constant operation constitutes a predicate lifting for T.

Let (C, γ) be a $T-$coalgebra and $M_\gamma = (C, R_\gamma, V_\gamma)$ the corresponding Kripke model. Now, it is time to show how the constant operation λ^p_C, defined above, helps us interpret propositional variables of the Kripke model M_γ. Since for any $U \subseteq C$,

$$\begin{array}{rcl}
(\gamma^{-1} \circ \lambda^p_C)(U) & = & \gamma^{-1}[\lambda^p_C(U)] \\
& = & \{x \in C \mid \gamma(x) \in \lambda^p_C(U)\} \\
\text{(by construction of } \lambda^p\text{)} & = & \{x \in C \mid p \in \pi_2(\gamma(x))\} \\
& = & \{x \in C \mid M_\gamma, x \Vdash p\} \\
\text{(by definition of the valuation)} & = & V_\gamma(p),
\end{array}$$

the constant map $\gamma^{-1} \circ \lambda^p_C$ gives us the set of worlds satisfying the propositional variable $p \in \Phi$. Consequently, for any $p \in \Phi$, $\gamma^{-1} \circ \lambda^p$ corresponds to valuation of p, $V(p)$, under the correspondence between $T-$coalgebras and Kripke models.

There is a more general principle underlying the construction of this constant predicate lifting $\{\lambda^p_C\}_C$. We show it in Prop. 4.1.7.

Let us define $1 = \{0\}$ then there is a unique surjection $!_X : X \longrightarrow 1$ for any set X.

Proposition 4.1.7 Let T be an endofunctor on **Set** and $A \subseteq T(1)$.Then, the operation $\lambda_C^A : \bar{\mathcal{P}}(C) \longrightarrow \bar{\mathcal{P}}(T(C))$, given by $\lambda_C^A(X) = \{\omega \in T(C) \mid T(!_C)(\omega) \in A\}$, defines a constant predicate lifting for T.

Proof: Let T be an endofunctor on **Set** and suppose that $A \subseteq T(1)$. Then, define an operation as follows: for any set C,

$$\begin{array}{lcl} \bar{\mathcal{P}}(C) & \xrightarrow{\lambda_C^A} & \bar{\mathcal{P}}(T(C)) \\ X & \longmapsto & \lambda_C^A(X) = \{\omega \in T(C) \mid T(!_C)(\omega) \in A\}. \end{array}$$

First, note that for any set X,

$$\lambda_C^A(X) \quad = \quad T(!_C)^{-1}[A] \qquad (*)$$

Then, let for any $X_1, X_2 \subseteq C$, $X_1 \subseteq X_2$. As

$$\begin{array}{lrcl} \text{by } (*) & \lambda_C^A(X_1) & = & T(!_C)^{-1}[A] \\ \text{by } (*) & & = & \lambda_C^A(X_2), \end{array}$$

we have $\lambda_C^A(X_1) \subseteq \lambda_C^A(X_2)$. Hence, order preservation trivially holds.

As for naturality, first consider the below diagram:

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{\quad T \quad} & \mathbf{Set} \\ \begin{array}{lll} C & & \\ f \downarrow & \searrow^{!_C} & \\ D & \underset{!_D}{\longrightarrow} & 1 \end{array} & & \begin{array}{lll} T(C) & & \\ T(f) \downarrow & \searrow^{T(!_C)} & \\ T(D) & \underset{T(!_D)}{\longrightarrow} & T(1) \end{array} \end{array}$$

Let $f : C \longrightarrow D$ then $!_C = !_D \circ f$ because there is a unique function from C to 1. Then, since T preserves compositions, we also have

$$
\begin{aligned}
T(!_C) &= T(!_D \circ f) \\
&= T(!_D) \circ T(f) \qquad (**)
\end{aligned}
$$

Now, let's show that below diagram commutes, i.e., $\bar{\mathcal{P}}(T(f)) \circ \lambda^A_D = \lambda^A_C \circ \bar{\mathcal{P}}(f)$.

$$
\begin{array}{ccc}
 & \xrightarrow{\bar{\mathcal{P}}} & \\
\mathbf{Set} & \Downarrow \lambda^A & \mathbf{Set} \\
 & \xrightarrow[\bar{\mathcal{P}} \circ T]{} &
\end{array}
$$

$$
\begin{array}{lclcl}
C & & \bar{\mathcal{P}}(C) & \xrightarrow{\lambda^A_C} & \bar{\mathcal{P}}(T(C)) \\
\downarrow f & & \bar{\mathcal{P}}(f) \uparrow & \circ & \uparrow \bar{\mathcal{P}}(T(f)) \\
D & & \bar{\mathcal{P}}(D) & \xrightarrow[\lambda^A_D]{} & \bar{\mathcal{P}}(T(D))
\end{array}
$$

Since for any $V \subseteq D$,

$$
\begin{array}{rcl}
(\bar{\mathcal{P}}(T(f)) \circ \lambda^A_D\ )(V) & = & (T(f))^{-1}\left[\lambda^A_D(V)\right] \\
 & = & \{\omega \in T(C) \mid T(f)(\omega) \in \lambda^A_D(V)\} \\
\text{(by } (*)\text{)} & = & \{\omega \in T(C) \mid T(f)(\omega) \in T(!_D)^{-1}[A]\} \\
\text{(by a Galois connection)} & = & \{\omega \in T(C) \mid T(!_D)(T(f)(\omega)) \in A\} \\
\text{(by } (**)\text{)} & = & \{\omega \in T(C) \mid T(!_D \circ f)(\omega) \in A\} \\
\text{(by } (**)\text{)} & = & \{\omega \in T(C) \mid T(!_C)(\omega) \in A\} \\
 & = & T(!_C)^{-1}[A] \\
\text{(since } f^{-1}[V] \subseteq C, \text{ by } (*)\text{)} & = & (\lambda^A_C(V))(f^{-1}[V]) \\
 & = & (\lambda^A_C\ \circ \bar{\mathcal{P}}(f))(V),
\end{array}
$$

naturality of λ^A is also satisfied. So, $\left\{\lambda^A_C\right\}_{C\ a\ set}$ is a predicate lifting for T.

Now, we want to show how the above lifting λ^A, for $A \subseteq T(1)$, (where T is an arbitrary endofunctor on **Set**) is a generalization of λ^p for a propositional variable $p \in \Phi$ (when we choose T as $T(X) = \mathcal{P}(X) \times \mathcal{P}(\Phi)$ for any set X).

Consider $T(\_) = \mathcal{P}(\_) \times \mathcal{P}(\Phi)$ where Φ is the set of propositional variables. Thus, $T(1) = \mathcal{P}(1) \times \mathcal{P}(\Phi)$.

$$\begin{array}{ccccccc} & & \mathbf{Set} & \xrightarrow{\mathcal{P}(\_) \times \mathcal{P}(\Phi)} & \mathbf{Set} & & \\ C & \xrightarrow{!_C} & 1 & & \mathcal{P}(C) \times \mathcal{P}(\Phi) & \xrightarrow{\mathcal{P}(!_C) \times 1_{\mathcal{P}(\Phi)}} & \mathcal{P}(1) \times \mathcal{P}(\Phi) \end{array}$$

Then, choose $A = \{(V, Q) \in \mathcal{P}(1) \times \mathcal{P}(\Phi) \mid p \in Q\}$. So, $A \subseteq T(1)$.

Now, we show that $\lambda_C^A = \lambda_C^p$. Recall that the family of maps $\{\lambda_C^p\}_{C \text{ a set}}$ defined by, for any set U,

$$\lambda_C^p(U) = \{(V, Q) \in \mathcal{P}(C) \times \mathcal{P}(\Phi) \mid p \in Q\}$$

is a predicate lifting for T. On the other hand, for any $U \subseteq C$,

$$\begin{array}{lrl} \text{(by } (*)\text{)} & \lambda_C^A(U) & = T(!_C)^{-1}[A] \\ & & = \{\omega \in TC \mid T(!_C)(\omega) \in A\} \\ & & = \{(V, Q) \in TC \mid (\mathcal{P}(!_C) \times 1_{\mathcal{P}(\Phi)})(V, Q) \in A \\ & & = \{(V, Q) \in TC \mid (!_C[V], Q) \in A\} \\ (!_C[V] = \emptyset \text{ if } V = \emptyset \text{ and } !_C[V] = 1 \text{ if } V \neq \emptyset) & & = \{(V, Q) \in TC \mid p \in Q\} \\ \text{(by construction of } \lambda^p\text{)} & & = \lambda_C^p(U) \end{array}$$

Hence, $\lambda_C^A = \lambda_C^p$ for any set C. As a result, proposition 4.1.7 generalizes λ^p. However, if we chose, for any $p \in \Phi$, $A = \{\{0\}\} \times \{Q \in \mathcal{P}(\Phi) \mid p \in Q\}$ then for any $U \subseteq C$,

$$
\begin{array}{llll}
\text{by } (*) & \lambda_C^A(U) & = & T(!_C)^{-1}[A] \\
 & & = & \{(V,Q) \in \mathcal{P}(C) \times \mathcal{P}(\Phi) \mid (\mathcal{P}(!_C) \times 1_{\mathcal{P}(\Phi)})(V,Q) \in A\} \\
 & & = & \{(V,Q) \in \mathcal{P}(C) \times \mathcal{P}(\Phi) \mid (!_C[V], Q) \in A\} \\
\text{by construction of } A & & = & \{(V,Q) \in \mathcal{P}(C) \times \mathcal{P}(\Phi) \mid !_C[V] = 1 \textit{ and } p \in Q\}
\end{array}
$$

So, for any $(V,Q) \in \lambda_C^A(U)$, $V \neq \emptyset$ and $p \in Q$ because otherwise if $V = \emptyset$, then $!_C[V] = \emptyset$.

As a consequence, for any $T-$coalgebra (C,δ), $\delta^{-1} \circ \lambda_C^A(U)$ would fail to give $V_\delta(p)$ in general since it does not include the dead end points of the corresponding Kripke model (i.e. the states with $R_\delta[x] = \emptyset$) at which $p \in \Phi$ is satisfied. Formally, since

$$
\begin{array}{lll}
\text{(by } (*)) \quad (\delta^{-1} \circ \lambda_C^A)(U) & = & \delta^{-1}[T(!_C)^{-1}[A]] \\
 & = & \{x \in C \mid \delta(x) \in T(!_C)^{-1}[A]\} \\
\text{(by a Galois connection)} & = & \{x \in C \mid T(!_C)(\delta(x)) \in A\} \\
 & = & \left\{x \in C \mid (\mathcal{P}(!_C) \times 1_{\mathcal{P}(\Phi)})(\delta(x)) \in A\right\} \\
 & = & \{x \in C \mid (!_C[\pi_1(\delta(x)], \pi_2(\delta(x))) \in A\} \\
\text{(by construction of } A) & = & \{x \in C \mid (!_C[\pi_1(\delta(x)] = \{0\} \textit{ and } p \in \pi_2(\delta(x))\} \\
 & = & \{x \in C \mid [\pi_1(\delta(x)] \neq \emptyset \textit{ and } p \in \pi_2(\delta(x))\} \\
 & = & \{x \in C \mid R_\delta[x] \neq \emptyset \textit{ and } \; x \Vdash p\},
\end{array}
$$

$(\delta^{-1} \circ \lambda_C^A)(U)$ could only give us the set of points with successor states where p is satisfied, but if there is a dead end point $x_1 \in C$ with $x_1 \Vdash p$ in the corresponding Kripke model, then $V_\delta(p) \neq (\delta^{-1} \circ \lambda_C^A)(U)$.

On the other hand, if we took $A = \{\emptyset\} \times \{Q \in \mathcal{P}(\Phi) \mid p \in Q\}$, for any $p \in \Phi$, then given a $T-$coalgebra (C,δ), we would obtain the set of dead end points (i.e. with no successor states) of the corresponding Kripke model where $p \in \Phi$ holds. Formally,

$$
\begin{aligned}
\text{(by } (*)) \quad (\delta^{-1} \circ \lambda_C^A)(U) &= \delta^{-1}[T(!_C)^{-1}[A]] \\
&= \{x \in C \mid \delta(x) \in T(!_C)^{-1}[A]\} \\
\text{(by a Galois connection)} &= \{x \in C \mid T(!_C)(\delta(x)) \in A\} \\
&= \big\{x \in C \mid (\mathcal{P}(!_C) \times 1_{\mathcal{P}(\Phi)})(\delta(x)) \in A\big\} \\
&= \{x \in C \mid (!_C[\pi_1(\delta(x))], \pi_2(\delta(x))) \in A\} \\
\text{(by construction of } A) &= \{x \in C \mid (!_C[\pi_1(\delta(x)] = \emptyset \textit{ and } p \in \pi_2(\delta(x))\} \\
&= \{x \in C \mid [\pi_1(\delta(x)] = \emptyset \textit{ and } p \in \pi_2(\delta(x))\} \\
&= \{x \in C \mid R_\delta[x] = \emptyset \textit{ and } \ x \Vdash p\}
\end{aligned}
$$

Thus, it would again not be sufficient to give $V_\delta(p)$ in general since it lacks of points with successor states at which p holds. Hence, A should exactly be chosen as $\mathcal{P}(1) \times \{Q \in \mathcal{P}(\Phi) \mid p \in Q\}$ to be able to give $V(p)$.

The above proposition provides us the semantics of the systems in a wide range, i.e., from Kripke models to the ones corresponding to an arbitrary endofunctor T on **Set**.

4.2 Modal Operators and Predicate Liftings

In classical modal logic, one often defines the diamond operator $\Diamond$ by putting $\Diamond\phi = \neg\Box\neg\phi$ for any modal formula ϕ. Now, we show that this can already be accomplished on the level of predicate liftings.

Proposition 4.2.1 Suppose that λ is a predicate lifting for T. Then, the family of operations $\lambda_C^\neg : \bar{\mathcal{P}}(C) \longrightarrow (\bar{\mathcal{P}} \circ T)(C)$, defined by $\lambda_C^\neg(X) = T(C) \setminus \lambda_C(C \setminus X)$ for any X, is a predicate lifting for T.

Proof: Assume that λ is a predicate lifting for T. Then, define a transformation as follows:

$$
\begin{array}{lcll}
\mathbf{Set} & \overset{\bar{\mathcal{P}}}{\longrightarrow} \Downarrow \lambda^{\neg} \underset{(\bar{\mathcal{P}} \circ T)}{\longrightarrow} & \mathbf{Set} & . \\
C & & \bar{\mathcal{P}}(C) \;\; \underset{\longrightarrow}{\lambda^{\neg}_C} & \bar{\mathcal{P}}(T(C)) \\
 & & X \;\; \longmapsto & \lambda^{\neg}_C(X) = T(C) \setminus \lambda_C(C \setminus X) \quad (\star)
\end{array}
$$

We claim that $\{\lambda^{\neg}_C\}_{C \ a \ set}$ constitutes a predicate lifting for T.

First, we want to show the naturality of $\lambda^{\neg}$. Let $f : C \longrightarrow D$ then it is enough to show that below diagram commutes.

$$
\begin{array}{lcccc}
\mathbf{Set} & \overset{\bar{\mathcal{P}}}{\longrightarrow} \Downarrow \lambda^{\neg} \underset{\bar{\mathcal{P}} \circ T}{\longrightarrow} & \mathbf{Set} & & \\
C & & \bar{\mathcal{P}}(C) & \overset{\lambda^{\neg}_C}{\longrightarrow} & \bar{\mathcal{P}}(T(C)) \\
\downarrow f & & \bar{\mathcal{P}}(f) \uparrow & \circ & \uparrow \bar{\mathcal{P}}(T(f)) \\
D & & \bar{\mathcal{P}}(D) & \underset{\lambda^{\neg}_D}{\longrightarrow} & \bar{\mathcal{P}}(T(D))
\end{array}
$$

Since $\{\lambda_C\}_{C \ a \ set}$ is a predicate lifting for T, the below diagram commutes, i.e.,

$$
\begin{array}{lcccc}
\mathbf{Set} & \overset{\bar{\mathcal{P}}}{\longrightarrow} \Downarrow \lambda \underset{\bar{\mathcal{P}} \circ T}{\longrightarrow} & \mathbf{Set} & & \\
C & & \bar{\mathcal{P}}(C) & \overset{\lambda_C}{\longrightarrow} & \bar{\mathcal{P}}(T(C)) \\
\downarrow f & & \bar{\mathcal{P}}(f) \uparrow & \circ & \uparrow \bar{\mathcal{P}}(T(f)) \\
D & & \bar{\mathcal{P}}(D) & \underset{\lambda_D}{\longrightarrow} & \bar{\mathcal{P}}(T(D))
\end{array}
$$

we have $\bar{\mathcal{P}}(T(f)) \circ \lambda_D = \lambda_C \circ \bar{\mathcal{P}}(f)$. $(\bullet)$

Because for any $V \in \bar{\mathcal{P}}(D)$,

$$
\begin{aligned}
(\bar{\mathcal{P}}(T(f)) \circ \lambda^{\neg}_D)(V) &= (T(f))^{-1}\left[\lambda^{\neg}_D(V)\right] \\
&= \left\{\omega \in T(C) \mid (T(f))(\omega) \in \lambda^{\neg}_D(V)\right\} \\
\text{(by construction of } \lambda^{\neg}) \quad &= \{\omega \in T(C) \mid (T(f))(\omega) \in T(D) \setminus \lambda_D(D \setminus V)\} \\
&= T(C) \setminus (T(f))^{-1}\left[\lambda_D(D \setminus V)\right] \\
\text{(by naturality of } \lambda)(\bullet) \quad &= T(C) \setminus (\lambda_C((f)^{-1}[D \setminus V]) \\
&= T(C) \setminus \lambda_C(C \setminus f^{-1}[V]) \\
\text{(by construction of } \lambda^{\neg}) \quad &= \lambda^{\neg}_C(f^{-1}[V]) \\
&= (\lambda^{\neg}_C \circ \bar{\mathcal{P}}(f))(V),
\end{aligned}
$$

naturality of $\lambda^{\neg}$ holds.

As for order preservation of $\lambda^{\neg}$, let $A, B \subseteq C$ be such that $A \subseteq B$ then

$$C \setminus B \subseteq C \setminus A. \quad (\star\star)$$

Because $\{\lambda_C\}_C$ is a predicate lifting for T, order preservation holds for λ, i.e., $\lambda_C(C \setminus B) \subseteq \lambda_C(C \setminus A) \subseteq T(C)$.Thus, since

$$
\begin{aligned}
\text{(by } (\star)) \quad \lambda^{\neg}_C(A) &= T(C) \setminus \lambda_C(C \setminus A) \\
\text{(by } (\star\star)) \quad &\subseteq T(C) \setminus \lambda_C(C \setminus B) \\
\text{(by } (\star)) \quad &= \lambda^{\neg}_C \,,
\end{aligned}
$$

order preservation is also satisfied for $\lambda^{\neg}$. As a result, $\{\lambda^{\neg}_{(C)}\}_C$ is a predicate lifting for T as well.

For the remainder of this section, Λ denotes a set of predicate liftings and we put $\Lambda^{\neg} = \left\{\lambda^{\neg} \mid \lambda \in \Lambda\right\}$.

4.3 Coalgebraic Semantics of the Modal Operators for Different Functors

In this section we give the coalgebraic semantics of boxed formulas for $\mathcal{P}-$coalgebras. We start with the power set functor.

Example 4.3.1 Let T be the power set functor $\mathcal{P} : \mathbf{Set} \longrightarrow \mathbf{Set}$.

Now, we define two natural transformations $\lambda^{\Box}, \lambda^{\Diamond} : \bar{\mathcal{P}} \longrightarrow \bar{\mathcal{P}} \circ \mathcal{P}$ by putting for every set X,

$$\begin{array}{lcl} \bar{\mathcal{P}}(X) & \xrightarrow{\lambda^{\Box}_X} & \bar{\mathcal{P}}(\mathcal{P}(X)) \\ U & \longmapsto & \lambda^{\Box}_X(U) = \mathcal{P}(U) \qquad (\star) \end{array}$$

and

$$\begin{array}{lcl} \bar{\mathcal{P}}(X) & \xrightarrow{\lambda^{\Diamond}_X} & \bar{\mathcal{P}}(\mathcal{P}(X)) \\ U & \longmapsto & \lambda^{\Diamond}_X(U) = \{V \subseteq X \mid V \cap U \neq \emptyset\}. \qquad (\star\star) \end{array}$$

Then, the coalgebraic semantics of boxed formulas $\left[\lambda^{\Box}\right]\varphi$ and $\left[\lambda^{\Diamond}\right]\varphi$ on a given $\mathcal{P}-$coalgebra $(X, \gamma : X \longrightarrow \mathcal{P}(X))$ is calculated as follows:

$$\begin{array}{llcl} \text{(by definition)} & \left[\left|\left[\lambda^{\Box}\right]\varphi\right|\right]_{(X,\gamma)} & = & \bar{\mathcal{P}}(\gamma)(\lambda^{\Box}_X([|\varphi|]_{(X,\gamma)})) \\ \text{(by } (\star)) & & = & \gamma^{-1}\left[\mathcal{P}([|\varphi|]_{(X,\gamma)})\right] \\ & & = & \left\{x \in X \mid \gamma(x) \in \mathcal{P}([|\varphi|]_{(X,\gamma)})\right\} \\ & & = & \left\{x \in X \mid R[x] \subseteq [|\varphi|]_{(X,\gamma)}\right\} \\ & & = & \left\{x \in X \mid \forall y (y \in R[x] \longrightarrow y \in [|\varphi|]_{(X,\gamma)})\right\} \\ \text{(by definition of } \ell_R) & & = & \ell_R([|\varphi|]_{(X,\gamma)}) \end{array}$$

and

$$
\begin{array}{lrcl}
\text{(by definition)} & \left[\left|\left[\lambda^{\Diamond}\right]\varphi\right|\right]_{(X,\gamma)} & = & \bar{\mathcal{P}}(\gamma)(\lambda^{\Diamond}_X([|\varphi|]_{(X,\gamma)})) \\
\text{(by } (\star\star)) & & = & \gamma^{-1}\left[\left\{V \subseteq X \mid V \cap [|\varphi|]_{(X,\gamma)} \neq \emptyset\right\}\right] \\
 & & = & \left\{x \in X \mid \gamma(x) \cap [|\varphi|]_{(X,\gamma)} \neq \emptyset\right\} \\
 & & = & \left\{x \in X \mid \exists y \in \gamma(x) \ni y \in [|\varphi|]_{(X,\gamma)}\right\}.
\end{array}
$$

where $\forall x \in X$, $\gamma(x) = R\left[x\right]$ is interpreted as the set of all successor states of x.

If we view the $\mathcal{P}-$coalgebra (X,γ) as a Kripke frame (X, R_γ) with

$$
x_1 R_\gamma x_2 :\Longleftrightarrow x_2 \in \gamma(x_1),
$$

we see that the coalgebraic semantics of the language given by $\Lambda = \left\{\lambda^{\Box}, \lambda^{\Diamond}\right\}$ coincides with the ordinary Kripke semantics in (X, R_γ) where for any modal formulas $\Box\varphi$ and $\Diamond\varphi$,

$$
\begin{array}{lrcl}
\text{(by definition)} & [|\Box\varphi|]_{(X,\gamma)} & = & \{x \in X \mid \forall y(xR_\gamma y \longrightarrow y \Vdash \varphi)\} \\
 & & = & \left\{x \in X \mid \forall y(y \in R_\gamma\left[x\right] \longrightarrow y \in [|\varphi|]_{(X,\gamma)})\right\} \\
 & & = & \left\{x \in X \mid R_\gamma\left[x\right] \subseteq [|\varphi|]_{(X,\gamma)}\right\} \\
 & & = & \left\{x \in X \mid R_\gamma\left[x\right] \in \mathcal{P}\left[|\varphi|\right]_{(X,\gamma)}\right\}
\end{array}
$$

and

$$
\begin{array}{lrcl}
\text{(by definition)} & [|\Diamond\varphi|]_{(X,\gamma)} & = & \{x \in X \mid \exists y \in X \ni xR_\gamma y \textit{ and } y \Vdash \varphi)\} \\
 & & = & \left\{x \in X \mid \exists y \in R_\gamma\left[x\right] \ni y \in [|\varphi|]_{(X,\gamma)}\right\} \\
 & & = & \left\{x \in X \mid R_\gamma\left[x\right] \cap [|\varphi|]_{(X,\gamma)} \neq \emptyset\right\}.
\end{array}
$$

Now, let's have a look at another example of predicate liftings.

Example 4.3.2 Let T be the functor $\mathcal{P}(\_) \times \mathcal{P}(\Phi) : \mathbf{Set} \longrightarrow \mathbf{Set}$ where Φ is the finite set (of propositional letters).

Now, we define two natural transformations $\lambda^1$and λ^2 :

$$\begin{array}{lcll} \mathbf{Set} & \overset{\bar{\mathcal{P}}}{\longrightarrow} \Downarrow \lambda^1 \underset{(\bar{\mathcal{P}} \circ T)}{\longrightarrow} & \mathbf{Set} & . \\ X & & \bar{\mathcal{P}}(X) \ \underset{\longrightarrow}{\lambda^1_X} & \bar{\mathcal{P}}(T(X)) \\ & & U \quad \longmapsto & \lambda^1_X(U) = \mathcal{P}(U) \times \mathcal{P}(\Phi) \quad (\star) \end{array}$$

and

$$\begin{array}{lcll} \mathbf{Set} & \overset{\bar{\mathcal{P}}}{\longrightarrow} \Downarrow \lambda^2 \underset{(\bar{\mathcal{P}} \circ T)}{\longrightarrow} & \mathbf{Set} & . \\ X & & \bar{\mathcal{P}}(X) \ \underset{\longrightarrow}{\lambda^2_X} & \bar{\mathcal{P}}(T(X)) \\ & & U \quad \longmapsto & \lambda^2_X(U) = \{(K, L) \in \mathcal{P}(X) \times \mathcal{P}(\Phi) \mid K \cap U \neq \emptyset\} \quad (\star\star) \end{array}$$

We claim that $\lambda^1$and λ^2 are predicate liftings for T.

Consider the below diagrams:

$$\begin{array}{lccc} \mathbf{Set} & \overset{\bar{\mathcal{P}}}{\longrightarrow} \Downarrow \lambda^1 \underset{\bar{\mathcal{P}} \circ T}{\longrightarrow} & \mathbf{Set} & \\ X & & \bar{\mathcal{P}}(X) \overset{\lambda^1_X}{\longrightarrow} & \bar{\mathcal{P}}(T(X)) \\ \downarrow f & & \bar{\mathcal{P}}(f) \uparrow \quad \circ & \uparrow \bar{\mathcal{P}}(T(f)) \\ Y & & \bar{\mathcal{P}}(Y) \underset{\lambda^1_Y}{\longrightarrow} & \bar{\mathcal{P}}(T(Y)) \end{array}$$

We have shown before that the above diagram commutes, i.e., $\lambda^1_X \circ \bar{\mathcal{P}}(f) = \bar{\mathcal{P}}(T(f)) \circ \lambda^1_Y$.

So, $\left\{\lambda^1_X : \bar{\mathcal{P}}(X) \longrightarrow \bar{\mathcal{P}}(\mathcal{P}(X) \times \mathcal{P}(\Phi))\right\}_X$ is a predicate lifting for T.

Now, we want to show that the following diagram also commutes i.e.$\lambda^2_X \circ \bar{\mathcal{P}}(f) = \bar{\mathcal{P}}(T(f)) \circ \lambda^2_Y$.

$$
\begin{array}{ccccc}
 & \overset{\bar{\mathcal{P}}}{\longrightarrow} & & & \\
\mathbf{Set} & \Downarrow \lambda^2 & \mathbf{Set} & & \\
 & \underset{\bar{\mathcal{P}}\circ T}{\longrightarrow} & & & \\
X & & \bar{\mathcal{P}}(X) & \overset{\lambda^2_X}{\longrightarrow} & \bar{\mathcal{P}}(T(X)) \\
\downarrow f & & \bar{\mathcal{P}}(f) \uparrow & \circ & \uparrow \bar{\mathcal{P}}(T(f)) \\
Y & & \bar{\mathcal{P}}(Y) & \underset{\lambda^2_Y}{\longrightarrow} & \bar{\mathcal{P}}(T(Y))
\end{array}
$$

Let $f : X \longrightarrow Y$ and $V \in \bar{\mathcal{P}}(Y)$. Then, since

$$
\begin{array}{rcl}
(\bar{\mathcal{P}}(T(f)) \circ \lambda^2_Y)(V) & = & (Tf)^{-1}\left[\lambda^2_Y(V)\right] \\
 & = & \{\omega \in T(X) \mid (Tf)(\omega) \in \lambda^2_Y(V)\} \\
 & = & \left\{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(\Phi) \mid (\mathcal{P}(f) \times 1_{\mathcal{P}(\Phi)})(A, B) \in \lambda^2_Y(V)\right\} \\
 & = & \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(\Phi) \mid (f\left[A\right], B) \in \lambda^2_Y(V)\} \\
\text{(by construction of } \lambda^2\text{)} & = & \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(\Phi) \mid f\left[A\right] \cap V \neq \emptyset\} \\
 & = & \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(\Phi) \mid A \cap f^{-1}\left[V\right] \neq \emptyset\} \\
\text{(by construction of } \lambda^2\text{)} & = & \lambda^2_X(f^{-1}\left[V\right]) \\
 & = & (\lambda^2_X \circ \bar{\mathcal{P}}(f))(V),
\end{array}
$$

naturality of λ^2 holds.

As for order preservation, for $A, B \subseteq X$ let $A \subseteq B$. Then, since

$$
\begin{array}{lrcl}
\text{(by } (\star\star)) & \lambda^2_X(A) & = & \{(K,L) \in \mathcal{P}(X) \times \mathcal{P}(\Phi) \mid K \cap A \neq \emptyset\} \\
 & & \subseteq & \{(K,L) \in \mathcal{P}(X) \times \mathcal{P}(\Phi) \mid K \cap B \neq \emptyset\} \\
\text{(by } (\star\star)) & & = & \lambda^2_X(B),
\end{array}
$$

order preservation holds.

As a result, $\left\{\lambda^2_X : \bar{\mathcal{P}}(X) \longrightarrow \bar{\mathcal{P}}(\mathcal{P}(X) \times \mathcal{P}(\Phi))\right\}_X$ is also a predicate lifting for T.

Now, let's define another transformation $\lambda^p :\longrightarrow \mathcal{P}(\_) \times \mathcal{P}(\Phi)$ by putting for any set X,

$$
\begin{array}{lcl}
\bar{\mathcal{P}}(X) & \xrightarrow{\lambda^p_X} & \bar{\mathcal{P}}(\mathcal{P}(X) \times \mathcal{P}(\Phi)) \\
U & \longmapsto & \lambda^p_X(U) = \{(K,L) \in \mathcal{P}(X) \times \mathcal{P}(\Phi) \mid p \in L\}.
\end{array}
$$

We have seen in Example 4.1.6 that $\{\lambda^p\}_{p\in\Phi}$ is a (constant) predicate lifting for T as well.

Now, consider a $T-$coalgebra $(X, \gamma : X \longrightarrow \mathcal{P}(X) \times \mathcal{P}(\Phi))$ where for any $x_1 \in X$,

$$
\gamma(x_1) = (R[x_1], Q[x_1]) \in \mathcal{P}(X) \times \mathcal{P}(\Phi)
$$

If we interpret $R[x_1]$ as the set of all successsor states of x_1 and $Q[x_1]$ the set of all propositional variables satisfied by x_1, we obtain a Kripke model $\mathcal{M}_\gamma = (X, R_\gamma, V_\gamma)$ with

$$
x_1 R_\gamma x_2 :\Longleftrightarrow x_2 \in R[x_1] = (\pi_1 \circ \gamma)(x_1) \qquad (\bullet)
$$

and for any $p \in \Phi$,

$$
x_1 \in V_\gamma(p) :\Longleftrightarrow p \in Q[x_1] = (\pi_2 \circ \gamma)(x_1). \quad (\bullet\bullet)
$$

Formally, we have

$$
\begin{array}{llcl}
\text{(by definition)} & (\bar{\mathcal{P}}(\gamma) \circ \lambda_X^p)(U) & = & \gamma^{-1}\left[\{(K, L) \in \mathcal{P}(X) \times \mathcal{P}(\Phi) \mid p \in L\}\right] \\
& & = & \{x \in X \mid p \in \pi_2 \circ \gamma(x)\} \\
& & = & \{x \in X \mid p \in Q[x]\} \\
& & = & V_\gamma(p).
\end{array}
$$

Hence, $\{\lambda^p\}_{p\in\Phi}$ gives us the propositional variables of the corresponding modal language and furthermore, they interpret the valuation V_γ of the Kripke model $\mathcal{M}_\gamma = (X, R_\gamma, V_\gamma)$.

On the other hand, given a Kripke model $\mathcal{M} = (W, R, V)$, we can also obtain a $T-$coalgebra $(W, \gamma : W \longrightarrow \mathcal{P}(W) \times \mathcal{P}(\Phi))$ by putting $\forall w \in W$,

$$
\gamma(w) = (R\,[w]\,, V^{'}\,[w]) \in \mathcal{P}(W) \times \mathcal{P}(\Phi)
$$

where $V^{'}\,[w] = \{p \in \Phi \mid w \in V(p)\}$ and $R\,[w] = \left\{w^{'} \in X \mid wRw^{'}\right\}$.

As a result, there is a one-to-one correspondence between the $T-$coalgebras and the corresponding Kripke models.

On the other hand, the coalgebraic semantics of boxed formulas $[\lambda^1]\,\varphi$ and $[\lambda^2]\,\psi$ on a given $T-$coalgebra (X, γ) is calculated as follows:

$$
\begin{array}{llcl}
\text{(by definition)} & [|[\lambda^1]\,\varphi|]_{(X,\gamma)} & = & (\bar{\mathcal{P}}(\gamma) \circ \lambda_X^1)([|\varphi|]_{(X,\gamma)}) \\
\text{(by construction of } \lambda^1) & & = & \gamma^{-1}\left[\mathcal{P}([|\varphi|]_{(X,\gamma)}) \times \mathcal{P}(\Phi)\right] \\
& & = & \left\{x \in X \mid \pi_1 \circ \gamma(x) \in \mathcal{P}([|\varphi|]_{(X,\gamma)}) \textit{ and } \pi_2 \circ \gamma(x) \in \mathcal{P}(\Phi)\right\} \\
\text{(by } (\bullet) \text{ and } (\bullet\bullet)) & & = & \left\{x \in X \mid R[x] \subseteq [|\varphi|]_{(X,\gamma)} \textit{ and } Q[x] \subseteq \Phi\right\} \\
& & = & \{x \in X \mid \forall y (x R_\gamma y \longrightarrow y \vDash_\gamma \varphi\} \\
\text{(by definition of } \ell_R) & & = & \ell_{R_\gamma}([|\varphi|]_{(X,\gamma)})
\end{array}
$$

and

$$
\begin{array}{llll}
\text{(by definition)} & [|[\lambda^2]\,\varphi|]_{(X,\gamma)} & = & (\bar{\mathcal{P}}(\gamma)\circ\lambda^2_X)([|\varphi|]_{(X,\gamma)}) \\
\text{(by construction of } \lambda^2) & & = & \gamma^{-1}\left[\left\{(K,L)\in\mathcal{P}(X)\times\mathcal{P}(\Phi)\mid K\cap[|\varphi|]_{(X,\gamma)}\neq\emptyset\right\}\right] \\
 & & = & \left\{x\in X\mid(\pi_1\circ\gamma(x))\cap[|\varphi|]_{(X,\gamma)}\neq\emptyset\;\; and\;\;(\pi_2\circ\gamma(x))\subseteq\Phi\right\} \\
\text{(by } (\bullet) \text{ and } (\bullet\bullet)) & & = & \left\{x\in X\mid R\,[x]\cap[|\varphi|]_{(X,\gamma)}\neq\emptyset\; and\; Q[x]\subseteq\Phi\right\} \\
 & & = & \left\{x\in X\mid\exists y\in R\,[x]\ni y\in[|\varphi|]_{(X,\gamma)}\right\} \\
 & & = & \{x\in X\mid\exists y\in X\ni xR_\gamma y\; and\; y\vDash_\gamma\varphi\} \\
\text{(by definition of } m_R) & & = & m_{R_\gamma}([|\varphi|]_{(X,\gamma)}).
\end{array}
$$

Hence, we have seen that the coalgebraic semantics of the language $\mathcal{L}(\Lambda)$ given by $\Lambda=\{\lambda^1,\lambda^2\}\cup\{\lambda^p\}_{p\in\Phi}$ coincides with the ordinary Kripke semantics.

Chapter 5

Correspondence Between $\mathcal{P}_\omega$-coalgebras and Image-finite Kripke Frames

In this section we show corresponding $T-$coalgebras to the image finite frames and observe which functor gives us these $T-$coalgebras. Then, associated with this functor, we give a set of predicate liftings for T and the coalgebraic modal logic given by this set of predicate liftings for T .

5.1 Predicate Liftings For $\mathcal{P}_\omega$

In this section first we introduce the suitable language of the coalgebraic modal logic for the finite power set functor, $\mathcal{P}_\omega$. Then we give the coalgebraic modal logic for $\mathcal{P}_\omega$ and the corresponding modal logic using predicate liftings.

Now, let's consider the finite power set functor $\mathcal{P}_\omega : \mathbf{Set} \longrightarrow \mathbf{Set}$, defined by for any set X,

$$\mathcal{P}_\omega(X) = \{V \subseteq X \mid |V| < \omega\} = \{V \subseteq X \mid V \textit{ is finite}\} . \quad (\bullet)$$

We define two transformations $\lambda^{\Box_\omega}, \lambda^{\Diamond_\omega} : \bar{\mathcal{P}} \longrightarrow \bar{\mathcal{P}} \circ \mathcal{P}_\omega$ by putting for every set X,

$$\begin{array}{rlcl} \lambda_X^{\Box_\omega} : & \bar{\mathcal{P}}(X) & \longrightarrow & \bar{\mathcal{P}}(\mathcal{P}_\omega(X)) \\ & U & \longmapsto & \lambda_X^{\Box_\omega}(U) = \{V \subseteq U \mid |V| < \omega\} = \mathcal{P}_\omega(U) \quad (\star) \end{array}$$

and

$$\begin{array}{rlcl} \lambda_X^{\Diamond_\omega} : & \bar{\mathcal{P}}X & \longrightarrow & \bar{\mathcal{P}}(\mathcal{P}_\omega(X)) \\ & U & \longmapsto & \lambda_X^{\Diamond_\omega}(U) = \{V \subseteq_\omega X \mid V \cap U \neq \emptyset\}. \quad (\star\star) \end{array}$$

$\lambda^{\Box_\omega}, \lambda^{\Diamond_\omega}$ are natural transformations as we have shown below.

Consider the following diagram:

$$\begin{array}{ccccc} & \xrightarrow{\bar{\mathcal{P}}} & & & \\ \mathbf{Set} & \Downarrow \lambda^{\Box_\omega} & \mathbf{Set} & & \\ & \xrightarrow[\bar{\mathcal{P}}\circ\mathcal{P}_\omega]{} & & & \\ X & & \bar{\mathcal{P}}(X) & \xrightarrow{\lambda_X^{\Box_\omega}} & \bar{\mathcal{P}}(\mathcal{P}_\omega(X)) \\ \downarrow f & & \bar{\mathcal{P}}(f) \uparrow & \circ & \uparrow \bar{\mathcal{P}}(\mathcal{P}_\omega(f)) \\ Y & & \bar{\mathcal{P}}(Y) & \xrightarrow[\lambda_Y^{\Box_\omega}]{} & \bar{\mathcal{P}}(\mathcal{P}_\omega(Y)) \end{array}$$

For any $f : X \longrightarrow Y$, we need to show that the above diagram commutes, i.e., $\lambda_X^{\Box_\omega} \circ \bar{\mathcal{P}}(f) = \bar{\mathcal{P}}(\mathcal{P}_\omega(f)) \circ \lambda_Y^{\Box_\omega}$. Since for any $E \subseteq Y$,

$$
\begin{array}{llll}
\text{(by } (\star)\text{)} & (\bar{\mathcal{P}}(\mathcal{P}_\omega(f) \circ \lambda_Y^{\Box_\omega})(E) & = & (\mathcal{P}_\omega(f))^{-1}\left[\mathcal{P}_\omega(E)\right] \\
 & & = & \{K \in \mathcal{P}_\omega(X) \mid \mathcal{P}_\omega(f)(K) \in \mathcal{P}_\omega(E)\} \\
\text{(by } (\bullet)\text{)} & & = & \{K \subseteq_\omega X \mid card(f\left[K\right]) < \omega \text{ and } f\left[K\right] \subseteq E\} \\
\text{(by a Galois connection)} & & = & \{K \subseteq_\omega X \mid K \subseteq \ f^{-1}\left[E\right]\} \\
\text{(by } (\bullet)\text{)} & & = & \mathcal{P}_\omega(f^{-1}\left[E\right]) \\
\text{(by } (\star)\text{)} & & = & \lambda_X^{\Box_\omega}(f^{-1}\left[E\right]) \\
 & & = & (\lambda_X^{\Box_\omega} \circ \bar{\mathcal{P}}(f))(E),
\end{array}
$$

naturality of $\lambda^{\Box_\omega}$ holds.

As for order preservation of $\lambda^{\Box_\omega}$, for any $A, B \subseteq X$, let $A \subseteq B$. Then, since

$$
\begin{array}{llll}
\text{(by } (\star)\text{)} & \lambda_X^{\Box_\omega}(A) & = & \mathcal{P}_\omega(A) \\
 & & \subseteq & \mathcal{P}_\omega(B) \\
\text{(by } (\star)\text{)} & & = & \lambda_X^{\Box_\omega}(B),
\end{array}
$$

order preservation is also satisfied. Hence, $\left\{\lambda_X^{\Box_\omega}\right\}_X$ is a predicate lifting for $\mathcal{P}_\omega$. Now, consider the below diagram:

$$
\begin{array}{lll}
\mathbf{Set} & \overset{\bar{\mathcal{P}}}{\longrightarrow} \Downarrow \lambda^{\Diamond_\omega} \underset{\bar{\mathcal{P}} \circ \mathcal{P}_\omega}{\longrightarrow} & \mathbf{Set}
\end{array}
$$

$$
\begin{array}{lllll}
X & & \bar{\mathcal{P}}(S) & \overset{\lambda_S^{\Diamond_\omega}}{\longrightarrow} & \bar{\mathcal{P}}(\mathcal{P}_\omega(S)) \\
\downarrow f & & \bar{\mathcal{P}}(f) \uparrow & \circ & \uparrow \bar{\mathcal{P}}(\mathcal{P}_\omega(f)) \\
Y & & \bar{\mathcal{P}}(T) & \underset{\lambda_T^{\Diamond_\omega}}{\longrightarrow} & \bar{\mathcal{P}}(\mathcal{P}_\omega(T))
\end{array}
$$

First, we want to show the naturality of $\lambda^{\Diamond_\omega}$. It is enough to show that $\bar{\mathcal{P}}(\mathcal{P}_\omega(f))\circ \lambda_T^{\Diamond_\omega} = \lambda_S^{\Diamond_\omega} \circ \bar{\mathcal{P}}(f)$ for any $f : S \longrightarrow T$. As for any $X \subseteq T$,

$$
\begin{array}{llcl}
\text{(by } (\star\star)) & (\bar{\mathcal{P}}(\mathcal{P}_\omega(f)) \circ \lambda_T^{\Diamond_\omega})(X) & = & (\mathcal{P}_\omega(f))^{-1}\left[\{Y \in \mathcal{P}_\omega(T) \mid Y \cap X \neq \emptyset\}\right] \\
\text{(by } (\bullet)) & & = & \{K \in \mathcal{P}_\omega(S) \mid card(f\left[K\right]) < \omega \ and \ f\left[K\right] \cap X \neq \emptyset\} \\
 & & = & \{K \subseteq_\omega S \mid f\left[K\right] \cap X \neq \emptyset\} \\
 & & = & \{K \subseteq_\omega S \mid \exists t \in K \ni f(t) \in X\} \\
 & & = & \{K \subseteq_\omega S \mid K \cap f^{-1}\left[X\right] \neq \emptyset\} \\
\text{(by } (\star\star)) & & = & \lambda_S^{\Diamond_\omega}(f^{-1}\left[X\right]) \\
 & & = & (\lambda_S^{\Diamond_\omega} \circ \bar{\mathcal{P}}(f))(X),
\end{array}
$$

naturality holds.

Next, we will verify that $\lambda^{\Diamond_\omega}$ preserves order. For $A, B \subseteq S$, let $A \subseteq B$ then since

$$
\begin{array}{llcl}
\text{(by } (\star\star)) & \lambda_S^{\Diamond_\omega}(A) & = & \{X \subseteq_\omega S \mid X \cap A \neq \emptyset\} \\
 & & \subseteq & \{X \subseteq_\omega S \mid X \cap B \neq \emptyset\} \\
\text{(by } (\star\star)) & & = & \lambda_S^{\Diamond_\omega}(A),
\end{array}
$$

order preservation is satisfied. As a result, $\left\{\lambda_X^{\Diamond_\omega}\right\}_X$ is also a predicate lifting for $\mathcal{P}_\omega$.

5.2 Interpretation of Modalities for $\mathcal{P}_\omega$

In this section first we give coalgebraic semantics for boxed formulas for a $\mathcal{P}_\omega$-coalgebra. Then we show relationship between $\mathcal{P}_\omega$-coalgebras and image-finite Kripke frames. Consider a $\mathcal{P}_\omega$−coalgebra $(X, \gamma : X \longrightarrow \mathcal{P}_\omega(X))$ by defining for any $x_1 \in X$, $\gamma(x_1) \in \mathcal{P}_\omega(X)$ as the set of all successor states. Also, choose $[|\varphi|]_{(X,\gamma)} \subseteq X$. Then, we have

$$\begin{array}{lrcl}
\text{(by }(\star)\text{)} & (\bar{\mathcal{P}}(\gamma)\circ\lambda_X^{\Box_\omega})([|\varphi|]_{(X,\gamma)}) & = & \gamma^{-1}\left[\mathcal{P}_\omega([|\varphi|]_{(X,\gamma)})\right] \\
 & & = & \left\{x\in X \mid \gamma(x)\in\mathcal{P}_\omega([|\varphi|]_{(X,\gamma)})\right\} \\
\text{(by }(\bullet)\text{)} & & = & \left\{x\in X \mid card(\gamma(x))<\omega \text{ and } \gamma(x)\subseteq[|\varphi|]_{(X,\gamma)}\right\} \\
 & & = & \{x\in X \mid card(\gamma(x))<\omega \text{ and } \forall y\in\gamma(x), y\vDash_\gamma\varphi\} \\
\text{(by definition)} & & = & [|\Box\varphi|]_{(X,\gamma)}
\end{array}$$

and

$$\begin{array}{lrcl}
\text{(by }(\star\star)\text{)} & (\bar{\mathcal{P}}(\gamma)\circ\lambda_X^{\Diamond_\omega})([|\varphi|]_{(X,\gamma)}) & = & \gamma^{-1}\left[\left\{V\subseteq_\omega X \mid V\cap[|\varphi|]_{(X,\gamma)}\neq\emptyset\right\}\right] \\
 & & = & \left\{x\in X \mid card(\gamma(x))<\omega \text{ and } \gamma(x)\cap[|\varphi|]_{(X,\gamma)}\neq\emptyset\right\} \\
 & & = & \left\{x\in X \mid card(\gamma(x))<\omega \text{ and } \exists y\in\gamma(x)\ni y\in[|\varphi|]_{(X,\gamma)}\right\} \\
 & & = & \{x\in X \mid card(\gamma(x))<\omega \text{ and } \exists y\in\gamma(x)\ni y\vDash_\gamma\varphi\} \\
\text{(by definition)} & & = & [|\Diamond\varphi|]_{(X,\gamma)}\,.
\end{array}$$

Then, the coalgebraic semantics of boxed formulas $\left[\lambda^{\Box_\omega}\right]\varphi$ and $\left[\lambda^{\Diamond_\omega}\right]\varphi$ on a given $\mathcal{P}_\omega-$coalgebra (X,γ) is calculated as follows:

$$\left[\left|\left[\lambda^{\Box_\omega}\right]\varphi\right|\right]_{(X,\gamma)} = (\bar{\mathcal{P}}(\gamma)\circ\lambda_X^{\Box_\omega})([|\varphi|]_{(X,\gamma)})$$

and

$$\left[\left|\left[\lambda^{\Diamond_\omega}\right]\varphi\right|\right]_{(X,\gamma)} = (\bar{\mathcal{P}}(\gamma)\circ\lambda_X^{\Diamond_\omega})([|\varphi|]_{(X,\gamma)}).$$

Now, consider an arbitrary $\mathcal{P}_\omega-$coalgebra $(X,\ \gamma : X \longrightarrow \mathcal{P}_\omega(X))$ and define $R_\gamma\subseteq X\times X$ as follows: $\forall x_1,x_2\in X$,

$$x_1R_\gamma x_2 :\Leftrightarrow x_2\in\gamma(x_1)$$

where $\gamma(x_1) \in \mathcal{P}_\omega(X)$. Hence, we obtain a Kripke frame (X, R_γ) where for any $x \in X$, $\gamma(x) \subseteq X$ is finite, i.e., the set of successor states $\gamma(x)$ for any $x \in X$ is finite. Therefore, (X, R_γ) is image-finite.

Consequently, for any $\mathcal{P}_\omega-$coalgebra, the corresponding Kripke frame is image-finite. [see Ex 3.4.5]Hence, any two models $\mathcal{M}$, $\mathcal{M}'$ of these Kripke frames are also image-finite.

On the other hand, given an image-finite Kripke frame (X, R), we obtain a $\mathcal{P}_\omega-$coalgebra $(X,\, \gamma_R : X \longrightarrow \mathcal{P}_\omega(X))$ by defining a transition structure $\gamma_R : X \longrightarrow \mathcal{P}_\omega(X)$ on X with $\forall x_1 \in X$,

$$\gamma(x_1) = R\,[x_1] = \{y \in X \mid x_1 R y\}$$

as the (finite) set of all successor states.

As a consequence, there is a one-to-one correspondence between $\mathcal{P}_\omega-$coalgebras and image-finite frames.

If we take $\Lambda = \left\{\lambda^{\Box_\omega}, \lambda^{\Diamond_\omega}\right\}$ then we see that the coalgebraic semantics of the language $\mathcal{L}(\Lambda)$ coincides with the ordinary Kripke semantics.

Conclusion

This thesis is mainly concerned with the coalgebraic modal logic based on the finite power set functor. It also deals with the properties of the Kripke frame corresponding to coalgebras of this functor and how the predicate lifting, constructed, of this functor generalizes the box operator of usual Kripke semantics.

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Özgeçmiş

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Vitae

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