ISTANBUL KULTUR UNIVERSITY★ INSTITUTE OF SCIENCE

HIGH DEGREE B-SPLINE SOLUTION FOR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM

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STATEMENT OF NON PLAGIARIS

In this study I acknowledge non plagiarist from any one.

Khaled Elfaituri.

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ABSTRACT

HIGH DEGREE B-SPLINE SOLUTION FOR SINGULARlY

PERTURBED BOUNDARY VALUE PROBLEM

Khaled Elfaituri

This study deals with the singularly perturbed boundary value problems. It is very active filed now a days, especially with improvement technology of the computer machine which is help us to do million and million of mathematical operations. The perturbation theory benefits from this improvement to solve the boundary value problems, this kind of a applications can help us to solve a lot of problems occur in many areas of engineering and applied mathematics such as fluid mechanics, quantum mechanics, optimal control, chemical reactor theory, aerodynamics, reaction-diffusion process, geophysics, heat transport problems with large Peclet number and Navier-Strokes flows with large Reynolds numbers etc.

Perturbation theory comprises mathematical methods that are used to find an approximation solution to a problem which cannot be solved exactly, by starting from the exact solution to a related problem. Perturbation theory is applicable if the problem at hand can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem.

The study focuses on the some methods that solved this kind of the problems, the new scheme was used to apply the high degree b-spline interpolation, the result compared with the published methods recently.

Keywords: Perturbation theory, B-spline Interpolation, Finite Deference Method, Shooting Method.

HIGH DEGREE B-SPLINE SOLUTION FOR SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM

1. Introduction

Singularly perturbed boundary value problems are very active filed now a days, this kind of an applications can help us to solve a lot of problems occur in many areas of engineering and applied mathematics such as fluid mechanics, quantum mechanic, optimal control, chemical reactor theory, aerodynamics, reaction-diffusion process, geophysics, heat transport problems with large Peclet number and Navier-Strokes flows with large Reynolds numbers etc. Mathematically, Perturbation theory comprises mathematical methods that are used to find an approximate solution to a problem that cannot be solved exactly, by starting from the exact solution of a related problem.

Perturbation theory has its roots in 17th century celestial mechanics, where the theory of epicycles was used to make small corrections to the predicted paths of planets. Curiously, it was the need for more and more epicycles that eventually lead to the Copernican revolution in the understanding of planetary orbits. The development of basic perturbation theory for differential equations was fairly complete by the middle of the 19th century. It was at that time that Charles Delaunay was studying the perturbative expansion for the Earth-Moon-Sun system, and discovered the so-called "problem of small denominators". Here, the denominator appearing in the *n*'th term of the perturbative expansion could become arbitrarily small, causing the *n*'th correction to be as large as or larger than the first-order correction. At the turn of the 20th century, this problem lead Henri Poincare to make one of the first deductions of the existence of chaos, or what is prosaically called the "butterfly effect": that even a very small perturbation can have a very large effect on a system.

Perturbation theory saw a particularly dramatic expansion and evolution with the arrival of quantum mechanics. Although perturbation theory was used in the semiclassical theory of the Bohr atom, the calculations were monstrously complicated, and subject to somewhat ambiguous interpretation. The discovery of Heisenberg's matrix mechanics allowed a vast simplification of the application of perturbation theory. Notable examples are the Stark effect and the Zeeman effect, which have a simple enough theory to be included in standard undergraduate textbooks in quantum mechanics. Other early applications include the fine structure and the hyperfine structure in the hydrogen atom.

In modern times, perturbation theory underlies almost all of quantum chemistry and quantum field theory. In chemistry, perturbation theory was used to obtain the first solutions for the helium atom. The earliest use of perturbation theory for molecular physics was the development of the linear combination of atomic orbital's molecular orbital method (LCAO-MO) by Ugo Fano and others in the 1930's.

In the middle of the 20'th century, Richard Feynman realized that the perturbative expansion could be given a dramatic and beautiful graphical representation in terms of what are now called Feynman diagrams. Although originally applied only in quantum field theory, such diagrams now find increasing use in any area where perturbative expansions are studied.

A partial resolution of the small-divisor problem was given by the statement of the KAM theorem in 1954. Developed by Andrey Kolmogorov, Vladimir Arnold and Jurgen Moser, this theorem stated the conditions under which a system of partial differential equations will have only mildly chaotic behavior under small perturbations.

In the late 20th century, broad dissatisfaction with perturbation theory in the quantum physics community, including not only the difficulty of going beyond second order in the expansion, but also questions about whether the perturbative expansion is even convergent, has lead to a strong interest in the area of non-perturbative analysis, that is, the study of exactly solvable models. See [18].

Physical problems that are position-dependent rather than time-dependent are often described in terms of differential equations with conditions imposed at more than one point. The general two-point boundary-value problems in involve a second-order differential equation of the form:

$$y'' = f(x, y, y'),\ a \le x \le b, \tag{1.1}$$

Together with the boundary conditions:

$$y(a) = \alpha,\ and\ \ y(b) = \beta \tag{1.2}$$

Most of the material concerning second-order boundary-value problems can be extended to problems with boundary conditions of the form:

$$\alpha_1 y(a) - \beta_1 y'(a) = \alpha,\ and\ \ \alpha_2 y(a) - \beta_2 y'(b) = \beta \tag{1.3}$$

Where $|\alpha_1| + |\beta_1| \ne 0$, and, $|\alpha_2| + |\beta_2| \ne 0$ but some of the techniques become quite complicated. The reader who is interested in problems of this type is advised to consider a book specializing in boundary-value problems, such as [15].

Perturbation theory is applicable if the problem at hand can be formulated by adding a "small" term to the mathematical description of the exactly solvable problem.

In this study, in the first section the two famous methods will be presented which are the finite differences methods and the shooting method to solve the boundary value problems. In the next section, the definitions of five b-spline basis and the interpolating by use these b-spline bases will be exhibited, then it is used to solve the boundary value problem and will be compared its results with the results that published in the famous applied journal. In the last section, the singularly perturbed boundary value problem definition and its applications will be dissection, after that the new scheme will be applied to help the fifth degree b-spline interpolation to solve the singularly perturbed boundary value problem. Finally, the results will be compared by published result in the famous applied journal.

2. Boundary value problem

2.1. Introduction

Physical problems that are position-dependent rather than time-dependent are often descried in terms of differential equations with conditions imposed at more than one point. The general two-point boundary-value problems in this study involve a second-order differential equation of the form:

$$y'' = f(x, y, y'), \quad a \le x \le b \tag{2.1}$$

Together with the boundary conditions

$$y(a) = \alpha \text{, and } y(b) = \beta . \tag{2.2}$$

In the following sections, many of the methods will be described. All of them interpolate the solutions of the equation of the form (2.1) with the conditions of the form (2.2). This kind of the problems occurs in many areas of engineering branches of the sciences and mathematics such as fluid mechanic, quantum mechanic, optimal control, chemical reactor theory, aerodynamics, reaction-diffusion process, geophysics, heat transport problems with large Peclet numbers and Navier-Strokes flows with large Reynolds numbers etc. Recently, a lot of people prefer an interpolating the solutions instead of not solving these problems, this is why this study have been considered to add as soon as possible some of these interpolating methods.

2.1.1. Theorem

Suppose the function f in the boundary-value problem

$$y'' = f(x, y, y''), \quad a \le x \le b, \quad y(a) = \alpha, \; y(b) = \beta . \tag{2.3}$$

is continuous on the set

$$D = \left\{ (x, y, y'') \middle| a \le x \le b, -\infty < y < \infty, -\infty < y' < \infty \right\}$$

and that $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial y'}$ are also continuous on D. If

1. $\frac{\partial f}{\partial y}(x, y, y') > 0$ for all $(x, y, y') \in D$, and

2. A constant M exists, with

$$\frac{\partial f}{\partial y}(x, y, y')| \leq M \text{ , for all } (x, y, y') \in D \text{ .}$$

Then the boundary-value problem has a unique solution. The proofs see [15].

2.1.2. Example

The boundary value problem considered in [17]

$$y'' + e^{-xy} + \sin y' = 0,\ 1 \leq x \leq 2,\ y(1) = y(2) = 0 \tag{2.4}$$

Has an exact solution $f(x, y, y') = -e^{-xy} - \sin y'$.

And since

$$\frac{\partial f}{\partial y}(x, y, y') = x e^{-xy} > 0 \text{, and } \left|\frac{\partial f}{\partial y'}(x, y, y')\right| = \left|-\cos y'\right| \leq 1$$

Then the problem has a unique solution.

2.1.3. Definition

When $f(x, y, y')$ can be expressed in the form:

$$f(x, y, y') = p(x)y' + q(x)y + r(x) \quad . \tag{2.5}$$

The differential equation:

$$y'' = f(x, y, y') \tag{2.6}$$

is called **linear differential equation,** which is accure quite often in practice problems.

2.1.4. Corollary

If the linear boundary problem:

$$y'' = p(x)y' + q(x)y + r(x)\ ,\ a \leq x \leq b\ ,\ y(a) = \alpha\ ,\ y(b) = \beta\ . \tag{2.7}$$

Satisfies:

1. $p(x), q(x), \text{ and } r(x)$ are continuous on [a, b].
2. $q(x) > 0$ on [a, b],

Then the problem has a unique solution.

The proof of this corollary problem is this, suppose that changing to the two initial condition value problems

$$y'' = p(x)y' + q(x)y + r(x)\,,\ a \le x \le b\,, \quad y(a) = \alpha\,,\ y'(a) = 0\,, \text{ and}$$

$$y'' = p(x)y' + q(x)y\,,\ a \le x \le b\,, y(a) = 0\,,\ y'(a) = 1\,.$$

By Lipschitz condition theorem [17] - p.263 - the two problems have a unique solution, for example $y_1(x), \text{ and } y_2(x)$ are the solutions, respectively.

Then:

$$y(x) = y_1(x) + \left(\beta - \frac{y_1(b)}{y_2(b)}\right) y_2(x) \tag{2.8}$$

is the unique solution of our boundary value problem, provided that $y_2(b) \neq 0$.

Graphically in the figure (2.1), its clear that the solution can be approximated by $y(x)$ which is our unique solution.

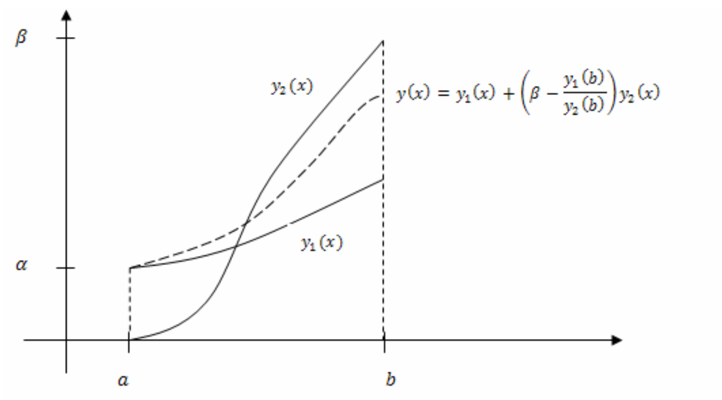

Fig. (2.1) shows uniqueness solution.

2.2 The Finite Difference Method

Consider the problem defined on the interval [a, b] of the form:

$$y'' = p(x)y' + q(x)y + r(x) \tag{2.9}$$

With the boundary conditions:

$$y(a) = \alpha, \text{ and } y(b) = \beta \quad . \tag{2.10}$$

Where $p(x), q(x), and\ r(x)$ are known smooth functions.

This kind of the problems (2.9) is a linear differential equation of second order which is clamped by boundary conditions (2.10). It is very important in many mathematical, physical sciences and engineering branches. In this section we will be focused on the solutions of this kind of the problems because we will study and compare the solution of these kinds of the problems by several methods later, finite difference method is one of them.

First of all, the interval will be divided to N subintervals, which length is $h = \frac{b-a}{N}$,

The interpolation solution is denoted by y_i for the exact solution $y(x_i)$, and from equation (2.10) denote that $y_0 = \alpha$, and $y_N = \beta$, the other interior nodes denotes by $y_1, y_2, y_3, \ldots, y_{N-1}$ that corresponding to the interior net points in the interval [a, b], The following graph in figure (1.2) is explaining that:

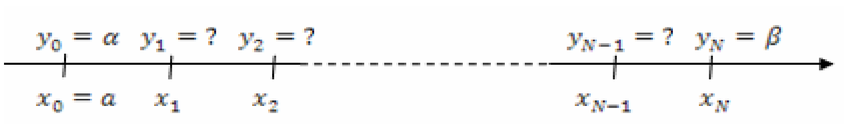

Fig. (2.2) shows boundary values and the unknown nodes

The equation (2.9) at $x = x_n$ leads to:

$$y''(x_n) = p(x_n)y'(x_n) + q(x_n)y(x_n) + r(x_n) \tag{2.11}$$

The simplest way to interpolate the equation (2.11) is replaced the differentiation $y''(x_n)$, and $y'(x_n)$ by its centered difference respectively which are:

$$y'(x_n) \cong \frac{y_{n+1} - y_{n-1}}{2h} \tag{2.12}$$

$$y''(x_n) \cong \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \tag{2.13}$$

Then substitute the equations (2.12) and (2.13) in the equation (2.11) leads to the form:

$$\left(\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \right) = p_n \left(\frac{y_{n+1} - y_{n-1}}{2h} \right) + q_n y_n + r_n \tag{2.14}$$

Where $r_n = r(x_n), p_n = p(x_n)$, and $q_n = q(x_n)$.

The equation (1.14) can be rewriting it as the form:

$$\left(1 + \frac{hp_n}{2}\right) y_{n-1} - \left(2 + h^2 q_n\right) y_n + \left(1 + \frac{hp_n}{2}\right) y_{n+1} = h^2 r_n \tag{2.15}$$

The equation (2.15) can be applying on the all interior nodes that belongs to [a, b] for n = 1, 2, …, N-1 respectively. Then, the system of the equations at the nodes is consisting of N-1 linear equations with N-1unknowns, which are $y_1, y_2, y_3, ..., y_{N-1}$. Because of $y_0 = \alpha$, and $y_N = \beta$, the first equation and the last equation leads to these forms respectively:

$$-\left(2 + h^2 q_1\right) y_1 + \left(1 - \frac{hp_1}{2}\right) y_2 = h^2 r_1 - \left(1 + \frac{hp_1}{2}\right)\alpha \tag{2.16}$$

$$\left(1 + \frac{hp_{N-1}}{2}\right) y_{N-2} - \left(2 + h^2 q_{N-1}\right) y_{N-1} = h^2 r_{N-1} - \left(1 - \frac{hp_{N-1}}{2}\right)\beta \tag{2.17}$$

Then the value of y_n where n = 1, 2, 3,…, N-1 can be calculated from the following system of the matrix form $A \cdot Y = C$ where A, Y, and C as the following form respectively:

$$A=\begin{bmatrix} -\left(2+h^2q_1\right) & \left(1-\frac{hp_1}{2}\right) & 0 & . & . & \cdots & 0 \\ \left(1+\frac{hp_2}{2}\right) & -\left(2+h^2q_2\right) & \left(1-\frac{hp_2}{2}\right) & 0 & . & \cdots & . \\ 0 & \left(1+\frac{hp_3}{2}\right) & -\left(2+h^2q_3\right) & \left(1-\frac{hp_3}{2}\right) & 0 & \cdots & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & \cdots & . & 0 & \left(1+\frac{hp_{N-2}}{2}\right) & -\left(2+h^2q_{N-2}\right) & \left(1-\frac{hp_{N-2}}{2}\right) \\ 0 & \cdots & . & . & 0 & \left(1+\frac{hp_{N-1}}{2}\right) & -\left(2+h^2q_{N-1}\right) \end{bmatrix} \tag{2.18}$$

$$Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ . \\ . \\ . \\ y_{N-2} \\ y_{N-1} \end{bmatrix}, \text{ and } C=\begin{bmatrix} h^2r_1-(1+\frac{h\,p_1}{2})\alpha \\ h^2r_2 \\ h^2r_3 \\ . \\ . \\ . \\ h^2r_{N-2} \\ h^2r_{N-1}-(1-\frac{h\,p_{N-1}}{2})\beta \end{bmatrix} \tag{2.19}$$

This system can be easily to solve. Then the unknowns which are denoted by vector Y will be known. These values are the interpolating values of y_i's by using the finite difference method. For more explanation, the next example will be demonstrating all the previous steps.

2.2.1 Example

Consider the problem:

$$y''-y=-4xe^x \tag{2.20}$$

With boundary conditions $y(0)=0$ and $y(1)=0$ defining on the interval [0, 1], whose exact solution is:

$$y=x(1-x)e^x\,. \tag{2.21}$$

Take h = 0.1, and then apply the system of the equations (2.18) & (2.19) to get:

$$A=\begin{bmatrix} -2.01 & 1 & 0 & . & . & ... & . & 0 \\ 1 & -2.01 & 1 & 0 & . & . & ... & . \\ 0 & 1 & -2.01 & 1 & 0 & . & ... & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & ... & . & . & 0 & 1 & -2.01 & 1 \\ 0 & ... & . & . & . & 0 & 1 & -2.01 \end{bmatrix}$$

And for Y and C are derived as the follows respectively:

$$Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ . \\ . \\ . \\ y_8 \\ y_9 \end{bmatrix}, \text{ and } C=\begin{bmatrix} -0.0044 \\ -0.0098 \\ -0.0162 \\ -0.0239 \\ -0.0330 \\ -0.0437 \\ -0.0564 \\ -0.0712 \\ -0.885 \end{bmatrix}.$$

This system of the form $Y = A^{-1} \cdot C$ can be solved to find the interpolating values of y_i's.

The result will be compared with the exact solution which is clear in the figure (2.3), the interpolating result by the finite difference method is denoted by (*).

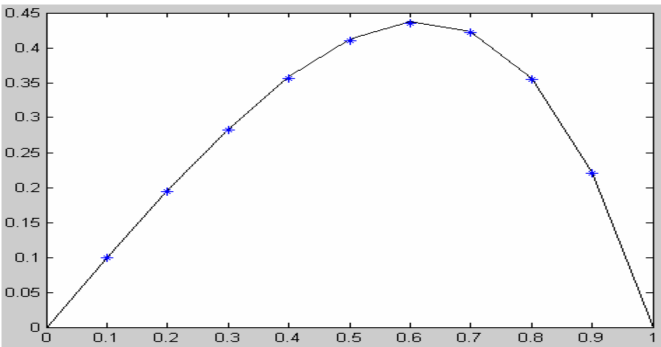

Fig. (2.3) shows exact function and I.F.D.M. by (*).

The table (2.1) gives the final result of our problem and the difference is clear between the exact solution and the finite difference method:

x_i	Exact solution y_i	F.D.M. $y(x_i)$	$\|y_i - y(x_i)\|$
0	0	0	---------
0.1	0.0995	0.0989	0.0006
0.2	0.1954	0.1944	0.0010
0.3	0.2853	0.2820	0.0014
0.4	0.3580	0.3563	0.0017
0.5	0.4122	0.4103	0.0019
0.6	0.4373	0.4354	0.0020
0.7	0.4229	0.4211	0.0018
0.8	0.3561	0.3546	0.0015
0.9	0.2214	0.2205	0.0009
1.0	0	0	-------

Table (2.1) shows the result of example (2.2.1), and its error.

2.2.2. Max. Error:

The maximum absolute error is 0.0020 which can be smaller than it if h is smaller than (0.1).

2.3. Shooting Method

The main idea of the shooting method is looks like the soldier who shoots a bomb to the target for example a castle and he try to change the coordinates of the mortar until towards to the target. The figure (2.4) explains this idea see [12].

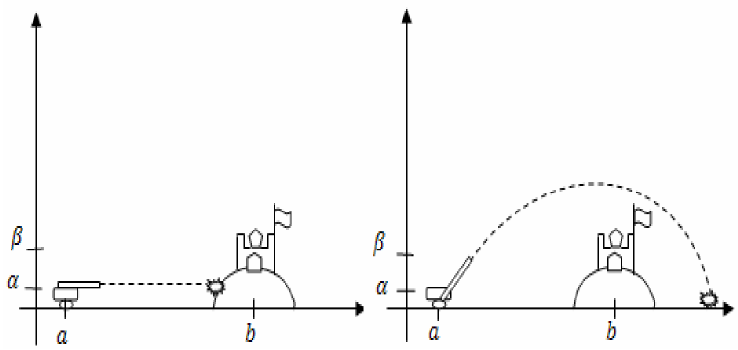

Fig. (2.4) shows the idea of shooting method

Now, we will explain the shooting method to solve the boundary value problem defined on (2.1) & (2.2). The main idea of this method is changing the boundary value problem to the initial value problem, and by help of guessing of the first derivative at the point $a \in [a,b]$ which is $y'(a)=\lambda$, the problem will be changed to the form:

$$y''=f(x,y,y'), \text{ with } y(a)=\alpha \text{ and } y'(a)=\lambda \quad (2.21)$$

Then the problem will be solved as initial value problem by chose any method for example Runge-Kutta method and the result be compared with exact value of $y(b)=\beta$. If the error is big enough then the value of $y'(a)=\lambda$ will be changed until the result is near as soon as possible to the exact value which is depending on the error of our hypotheses. The next example will be showing these steps practically.

2.3.1. Example

Let $y''-3y'+2y=0$ with clamped condition $y(0)=0$ and $y(1)=2$, defined on $[0,1]$. The exact solution of this problem is:

$$y(x)=\frac{2}{e-e^2}(e^x-e^{2x}) \quad (2.22)$$

First of all, the problem will be changed to the first order of two differential equations to decrease the order as this form:

$$u'=v \text{ , and } v'=3v-2u \quad (2.23)$$

With two conditions $u(0)=0$ and $v(0)=\lambda$, then our problem leads to the initial value problem and then the shooting method can be apply with the target $u(1)=2$.

Now, there is many methods can be apply to solve this problem, the Runge-Kutta with forth steps will be chosen with $h=0.1$. Fore a step n the forth steps Runge-Kutta method as the followed form:

$k_1=f(x_n,u_n,v_n)$, $l_1=g(x_n,u_n,v_n)$.

$$k_2 = f\,(x_n + \frac{h}{2}, u_n + \frac{h}{2}k_1, v_n + \frac{h}{2}l_1),\ l_2 = g\,(x_n + \frac{h}{2}, u_n + \frac{h}{2}k_1, v_n + \frac{h}{2}l_1)$$

$$k_3 = f\,(x_n + \frac{h}{2}, u_n + \frac{h}{2}k_2, v_n + \frac{h}{2}l_2),\ l_3 = g\,(x_n + \frac{h}{2}, u_n + \frac{h}{2}k_2, v_n + \frac{h}{2}l_2).$$

$$k_4 = f\,(x_n + h, u_n + h\,k_3, v_n + h\,l_3),\ l_4 = g\,(x_n + h, u_n + h\,k_3, v_n + h\,l_3).$$

$$u_{n+1} = u_n + \frac{h}{6}[k_1 + 2k_2 + 2k_3 + k_4],\ v_{n+1} = v_n + \frac{h}{6}[l_1 + 2l_2 + 2l_3 + l_4].$$

Now, we construct a matlab program (kutta4.m) see [appendix] to do these steps starting by the initial value of $v_1 = \lambda = 0.5$, and with help of matlab programs that define the input of two functions $f\,(x_n, u_n, v_n)$ and $g\,(x_n, u_n, v_n)$ which gave the output u' and v' functions which comes from equation (2.23) respectively. Next, the following looping is necessary to get the results for all interior nodes which belong to the interval [0, 1]:

$h = 0.1,\ x_1 = 0,\ u_1 = 0$ *and* $v_1 = 0.5$.

for $i = 1:10$

$[u_{i+1}, v_{i+1}] = kutta4(x_i, u_i, v_i, h)$.

end.

This looping is derive the u_i's and v_i's which are the interpolating value of exact values y_i's and y_i''s respectively. The table (1.2) shows this result starting with $v_1 = 0.5$, and tries to improve our choice until we get the target exact result which is $u_{10} = 2$. By this shooting method, not only found the values of our problem but also the derivative at the nodes will be found too.

Denote in the first column which is containing the values of $v_i = \lambda$, the changing of these value leads to the changing of the last column which is containing the value of $u_{10} = y\,(1) = 2$. And also the last raw gives the interpolating of the values of $y_i(x_i)$

for the nodes $x_i \in [0,1]$, where i = 0, 1, …, 10. The maximum error of this problem is:

$$\text{Error} = \max \| y_i - u_i \| = 4.5013\text{e-}005.$$

The figure (2.5) shows the exact solution and the solution of example (2.3.1) by shooting method denoted by (*) which is in the last raw of table (2.2).

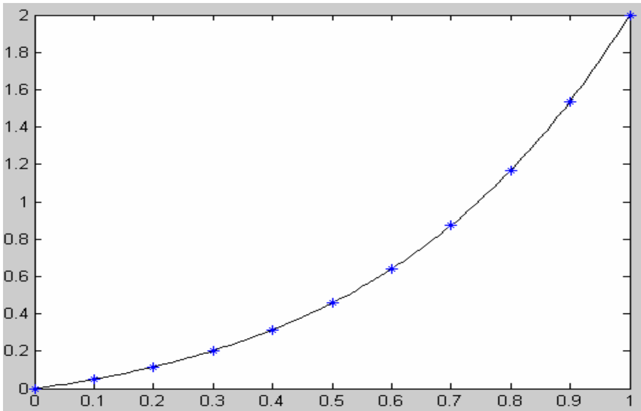

Fig. (2.5) shows the result of example (2.3.1)

$v_i = \lambda$	u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}
0.5	0	0.0581	0.1352	0.2361	0.3668	0.5348	0.7490	1.0207	1.3637	1.7950	2.3353
0.4	0	0.0455	0.1082	0.1889	0.2935	0.4278	0.5992	0.8166	1.0910	1.4360	1.8682
0.44	0	0.0511	0.1190	0.2078	0.3228	0.4705	0.6591	0.8982	1.2001	1.5796	2.0551
0.427	0	0.0496	0.1155	0.2016	0.3133	0.4567	0.6396	0.8717	1.1646	1.5329	1.9944
0.4285	0	0.0498	0.1159	0.2024	0.3144	0.4583	0.6419	0.8747	1.1687	1.5383	2.0014
0.4281	0	0.0498	0.1158	0.2022	0.3141	0.4579	0.6413	0.8739	1.1676	1.5368	1.9995
0.4283	0	0.0498	0.1158	0.2023	0.3142	0.4581	0.6416	0.8743	1.1681	1.5376	2.0004
0.4282	0	0.0498	0.1158	0.2022	0.3142	0.4580	0.6414	0.8741	1.1679	1.5372	20000

Table (2.2) shows the result of example (2.3.1).

3. B-spline Interpolation

3.1 Introduction

The approach involves using the so called B-splines as a basis function. These are so named because of their use as basis function, but also because of their characteristic bell shape. Such curves are consistent with a spline approach in that their value and their first and second derivatives would have continuity at their extremes. Thus, continuity of $f(x)$ and its lower derivatives at the nodes is ensured. See [4].

In this section will be focus on the b-spline basis and their definitions. Next, the interpolation theory by b-spline bassis considered, So the b-spline have been used as a basis to interpolate the difficult function. Finally the results will be compared with other methods.

3.2 Piecewise Polynomial

Let $[a, b] \subset R$ be a finite interval and we introduce a set of partition $\Omega_n = \{x_0, x_1, x_2, ..., x_n\}$ of $[a, b]$, where x_i $(i = 0, 1, 2, 3, ..., n)$ are called the nodes of the partition as shown in figure (3.1).

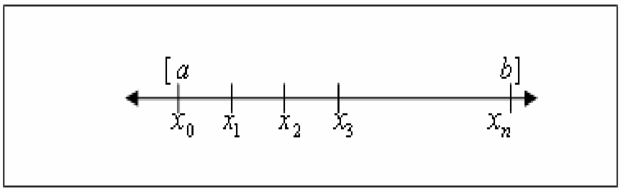

Fig. (3.1) shows nodes of partition

3.2.1. Definition

The set of piecewise polynomial of degree k defined on a partition $\Omega_n = \{x_0, x_1, x_2, ..., x_n\}$ denoted by $P_k(\Omega_n)$. A function belonging to $P_k(\Omega_n)$ in each subinterval $I_i = [x_{i=1}, x_i]$ is a k-th degree polynomial. Thus a piecewise polynomial of degree one see figure (3.2) is a function consisting of piecewise straight line segments.

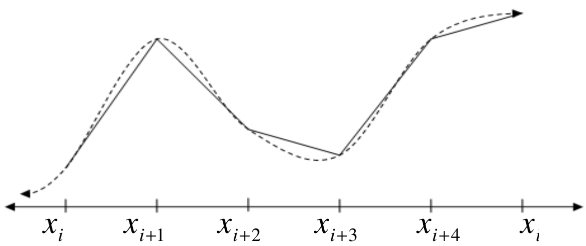

Fig. (3.2) shows a piecewise polynomial of degree one

3.3. The interpolation theory and B-spline

The interpolation theory is significant in many engineering fields especially those concerning applied mathematics such as chemical reactor theory, aerodynamics, quantum mechanics, optimal control, reaction-diffusion process, geophysics, ect.

The B-spline is chosen to apply the interpolation theory for these reasons:

- It can change a function of the difficult structure by a linear combination of simpler polynomial.
- The polynomial interpolation is one of the best methods used in practice, because of simplicity, differentiation, and calculating of its zeros.
- The approximations are piecewise polynomial of low degree, which is easily constructed, and the individual parts are smoothly connected.
- The B-spline functions constitute a very active field in the approximation theory because of using the boundary value conditions.
- The approximation converges and produces accurate results for a large class of function.

3.3.1. Statement of the problem of interpolation

On an interval [a, b] is specified (n+ 1) point $x_0, x_1, x_2, ..., x_n$, called nodes (mesh points or interpolation points), and the value of some function $f(x)$ at these points:

$$f(x_0) = y_0,\ f(x_1) = y_1,\ ....,f(x_n) = y_n \qquad (3.1)$$

It is required to construct a function $F(x)$ (interpolating function) belonging to a known class and assuming the same value at the interpolation points as $f(x)$, that is, such that

$$F(x_0) = y_0,\ F(x_1) = y_1,\ ....,F(x_n) = y_n \qquad (3.2)$$

Geometrically, this means that one has to find a curve $y = F(x)$ of some specific type that passes through the given set of points $M_i(x_i,\ y_i)$ *where* $i = 0, 1, 2, 3, ..., n$,

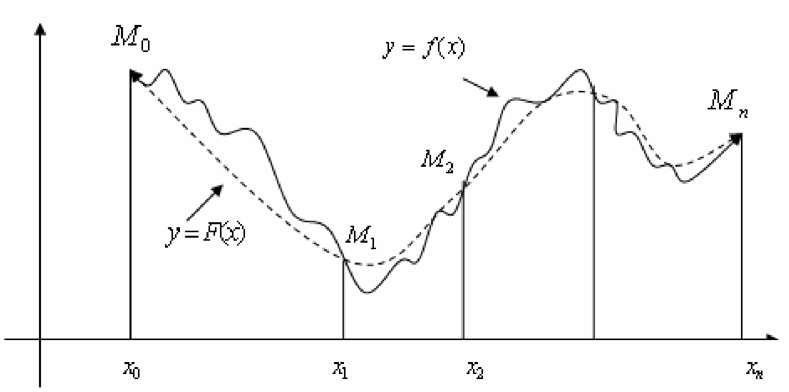

Fig.(3.3) Shows the nodes on the curve

In such a general statement, the problem can have infinity of solutions or none at all. However, the problem becomes unambiguous if in place of arbitrary function $F(x)$ we seek a polynomial $P_n(x)$ of degree not higher than n that satisfies the condition (2.2); that is, such that

$$P_n(x_0) = y_0, \qquad P_n(x_1) = y_1, \quad ....., P_n(x_n) = y_n \tag{3.3}$$

The resulting interpolation formula $y = F(x)$ is ordinarily used to approximate the values of the given function $f(x)$ for values of the argument x that differ from the interpolation points. This operation is called ***interpolation*** in the narrow sense when $x \in [x_0\ , x_n]$, that is the value of x is intermediate between x_0 *and* x_n, and ***extrapolation***, when $x \notin [x_0\ , x_n]$. See [10].

Note:

Specifically, we will use the equidistant partitions for all this study. Moreover, we extend the set of nodes in the interval [a, b] by taking

$$h = \frac{b-a}{n}\ , \quad x_0 = a, \text{ and } x_i = x_0 + ih, \text{ where } i = 1, 2, 3, ..., n.$$

3.3.2. B-spline Basis

The theory of spline function is very active field of approximation theory and boundary value problems (BVPs), when numerical aspects are considered.

A series of papers by [5-7] the BVPs of order third, fourth and fifth were solved using fourth and sixth-degree B-splines is very interesting to study. In the present study, several of different kind of B-spline interpolation degrees, definitions and applications, are focused to circulate the benefit. We will introduce two equivalent definitions for B-spline basis (Implicit, and Explicit definition).

3.3.2.1. Implicit definition

Let $\{\Omega_n\}$be a partition of $[a,b]\subset R$. A B-spline of order l is a spline from $S_l(\Omega_n)$ with minimal support and the partition of unity holding.

To explain this, let us defined $B_{i,j}(x)$ *where* $i\in Z$ is a B-spline of degree l, the left end of which support is equal to x_i, and then we have the following properties:

1. $\text{Supp}(B_{l,i})=[x_i\,,\,x_{i+l+1}]$ (3.4)

2. $B_{l,i}(x)\geq 0 \quad , \forall\ x\in R$ (Non-negativity) (3.5)

3. $\sum_{i=-\infty}^{\infty} B_{l,i}(x)=1 \quad , \forall\ x\in R$ (Partition of unity) (3.6)

The proof of these property see [9] p.131, and[1].

3.3.2.2. Explicit definition (recursion):

Let $\{\Omega_n\}$ be a set of partitions of $[a,b]\subset R$, we define the following basis:

1. The zero degree B-spline basis figure (3.4) is defined as:

$$B_{0,i}(x)=\begin{cases}1, & if\ x\in[x_i,x_{i+1}].\\ 0, & otherwise.\end{cases} \tag{3.7}$$

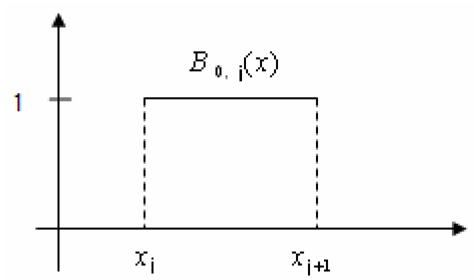

Fig.(3.4) shows $B_{0,1}(x)$ basis.

For a positive l we define the following recursion:

$$B_{l,i}(x)=\left(\frac{x-x_i}{x_{l+i}-x_i}\right)B_{l-1,i}(x)+\left(\frac{x_{l+i+1}-x}{x_{l+i+1}-x_{i+1}}\right)B_{l-1,i+1}(x) \qquad (3.8)$$

Then by (3.7) and recursion (3.8) the important high degree B-spline basis can be defend as follows:

2. The first degree B-spline basis defined as:

$$B_{1,i}(x)=\begin{cases}\dfrac{(x-x_i)}{(x_{i+1}-x_i)}, & if \;\; x\in[x_i,x_{i+1}].\\ \dfrac{(x_{i+2}-x)}{(x_{i+2}-x_{i+1})}, & if \;\; x\in[x_{i+1},x_{i+2}].\\ 0, & otherwise.\end{cases} \qquad (3.9)$$

Figure (3.5) illustrates the graph of the first degree b-spline basis (3.9), the properties (3.4), (3.5), and (3.6) are clear from the graph.

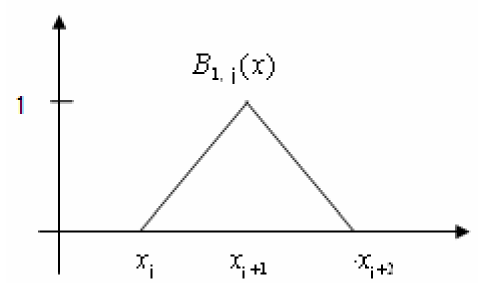

Fig.(3.5) shows $B_{1,i}(x)$ basis.

3. The second degree B-spline basis as the following form:

$$B_{2,i}(x) = \frac{1}{2h^2}\begin{cases}(x-x_i)^2, & if \ \ x \in [x_i, x_{i+1}].\\ h^2+2h(x-x_{i+1})-2(x-x_{i+1})^2, & if \ \ x \in [x_{i+1}, x_{i+2}].\\ (x_{i+3}-x)^2, & if \ \ x \in [x_{i+2}, x_{i+3}].\\ 0, & otherwise.\end{cases} \tag{3.10}$$

Figure (3.6) illustrates the second degree B-spline basis (3.10) as the follows:

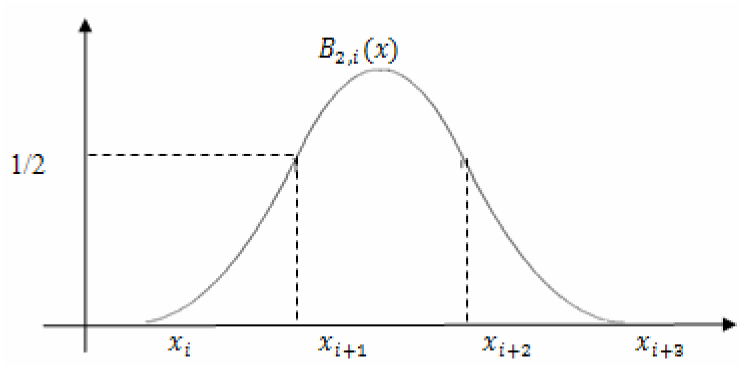

Fig. (3.6) illustrates $B_{2,i}(x)$ basis

The values and their derivatives of the second degree b-spline basis at the nodes are derived from the general form of the basis (3.10) by substitute the value of x by x_i , The table (2.1) is illustrate all this values which is very important when the interpolation method have been applied practically.

	x_i	x_{i+1}	x_{i+2}	x_{i+3}
$B_{2,i}(x)$	0	1/2	1/2	0
$B'_{2,i}(x)$	0	1/h	-1/h	0

Table (3.1) illustrate the values of $B_{2,i}$'s and their derivatives

4. The third degree B-spline basis $B_{3,i}(x)$ is very important; because in the next sections we will be use this basis to solve the high degree problems. The form of third degree b-spline as the following form:

$$B_{3,i}(x)=\frac{1}{6h^3}\begin{cases}(x-x_i)^3, & if \;\; x\in[x_i,x_{i+1}].\\ h^3+3h^2(x-x_{i+1})+3h(x-x_{i+1})^2\ldots & \\ \qquad -3(x-x_{i+1})^3, & if \;\; x\in[x_{i+1},x_{i+2}].\\ h^3+3h^2(x_{i+3}-x)+3h(x_{i+3}-x)^2\ldots & \\ \qquad -3(x_{i+3}-x)^3, & if \;\; x\in[x_{i+3},x_{i+4}].\\ (x_{i+4}-x)^3 & if \;\; x\in[x_{i+4},x_{i+5}].\\ 0, & otherwise.\end{cases} \tag{3.11}$$

The Figure (3.7) shows the third degree b-spline basis and the property (3.4), (3.5), and (3.6) are clear in the graph.

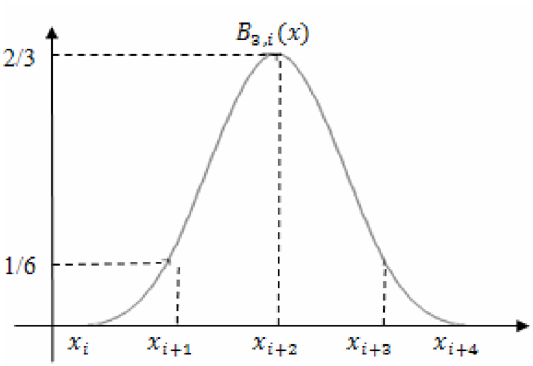

Fig. (3.7) shows $B_{3,i}(x)$ basis

The value of the third degree b-spline basis at the nodes and their derivatives are easy to fit from the basis (3.11), the table (3.2) shows these values:

	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}
$B_{3,i}(x)$	0	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$	0
$B'_{3,i}(x)$	0	$\frac{1}{2h}$	0	$-\frac{1}{2h}$	0
$B''_{3,i}(x)$	0	$\frac{1}{h^2}$	$\frac{-2}{h^2}$	$\frac{1}{h^2}$	0

Table (3.2) shows the values of $B_{3,i}$'s and their derivatives.

5. The fourth degree B-spline basis defined in the equation (3.13),

$$B_{4,i}(x)=\frac{1}{24h^4}\begin{cases}(x-x_i)^4, & if \;\; x\in[x_i,x_{i+1}].\\ h^4+4h^3(x-x_{i+1})+6h^2(x-x_{i+1})^2...\\ \quad +4h(x-x_{i+1})^3-4(x-x_{i+1})^4, & if \;\; x\in[x_{i+1},x_{i+2}].\\ 11h^4+12h^3(x-x_{i+2})-6h^2(x-x_{i+2})^2...\\ \quad -12h(x-x_{i+2})^3+6(x-x_{i+2})^4, & if \;\; x\in[x_{i+2},x_{i+3}].\\ h^4+4h^3(x_{i+4}-x)+6h^2(x_{i+4}-x)^2...\\ \quad +4h(x_{i+4}-x)^3-4(x_{i+4}-x)^4, & if \;\; x\in[x_{i+3},x_{i+4}].\\ (x_{i+5}-x)^4, & if \;\; x\in[x_{i+4},x_{i+5}].\\ 0, & otherwise.\end{cases} \qquad (3.12)$$

The value of the b-spline at the nodes and its derivative are considered in the following table:

	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}	x_{i+5}
$B_{4,i}(x)$	0	$\frac{1}{24}$	$\frac{11}{24}$	$\frac{11}{24}$	$\frac{1}{24}$	0
$B'_{4,i}(x)$	0	$\frac{1}{6h}$	$\frac{1}{2h}$	$-\frac{1}{2h}$	$-\frac{1}{6h}$	0
$B''_{4,i}(x)$	0	$\frac{1}{2h^2}$	$-\frac{1}{2h^2}$	$-\frac{1}{2h^2}$	$\frac{1}{2h^2}$	0

Table (3.3) shows the values of $B_{4,i}(x)$'s and their derivatives.

The figure (3.8) shows the fourth degree B-spline function and the corresponding value of the nodes which is clear the sum of these values equal to one (partition of unity property three equation (3.6)).

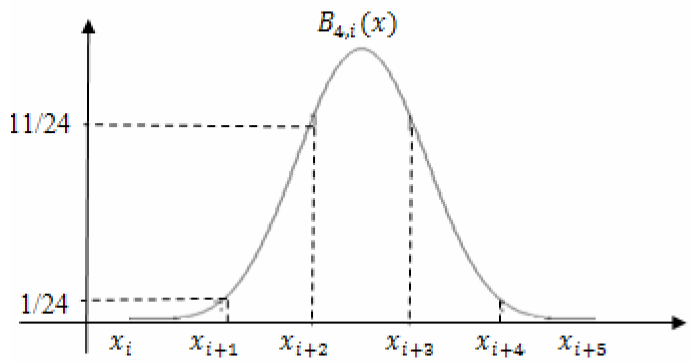

Fig. (3.8) shows $B_{4,i}(x)$ basis

6. The fifth degree B-spline basis defined as:

$$B_{5,i}(x)=\frac{1}{120h^5}\begin{cases}(x-x_i)^5, & if\ \ x\in[x_i,x_{i+1}].\\ h^5+5h^4(x-x_{i+1})+10h^3(x-x_{i+1})^2+10h^2(x-x_{i+1})^3\ldots \\ \quad +5h(x-x_{i+1})^4-5(x-x_{i+1})^5, & if\ \ x\in[x_{i+1},x_{i+2}].\\ 26h^5+50h^4(x-x_{i+2})+20h^3(x-x_{i+2})^2-20h^3(x-x_{i+2})^3\ldots \\ \quad -20h(x-x_{i+2})^4+10(x-x_{i+2})^5, & if\ \ x\in[x_{i+2},x_{i+3}].\\ 26h^5+50h^4(x_{i+4}-x)+20h^3(x_{i+4}-x)^2-20h^3(x_{i+4}-x)^3\ldots \\ -20h(x_{i+4}-x)^4+10(x_{i+4}-x)^5, & if\ \ x\in[x_{i+3},x_{i+4}].\\ h^5+5h^4(x_{i+5}-x)+10h^3(x_{i+5}-x)^2+10h^2(x_{i+5}-x)^3\ldots \\ \quad +5h(x_{i+5}-x)^4-5(x_{i+5}-x)^5, & if\ \ x\in[x_{i+4},x_{i+5}].\\ (x_{i+6}-x)^5, & if\ \ x\in[x_{i+5},x_{i+6}].\\ 0, & otherwise.\end{cases} \quad (3.13)$$

The graph of $B_{5,i}(x)$ basis is very important in the next sections; because we will be apply this basis to solve the high degree b-spline problems. Figure (3.9) shows the graph of this basis.

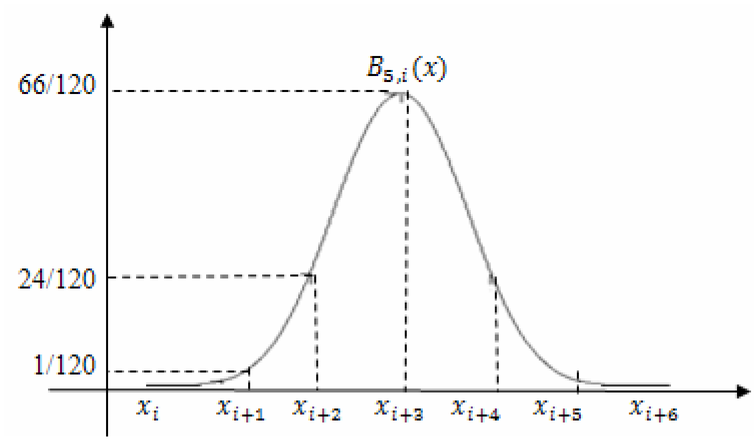

Fig. (3.9) shows $B_{5,i}(x)$ basis

The value and the derivatives of the fifth degree b-spline basis at the nodes are listed in the following table (3.4):

	x_i	x_{i+1}	x_{i+2}	x_{i+3}	x_{i+4}	x_{i+5}	x_{i+6}
$B_{5,i}(x)$	0	$\frac{1}{120}$	$\frac{26}{120}$	$\frac{66}{120}$	$\frac{26}{120}$	$\frac{1}{120}$	0
$B'_{5,i}(x)$	0	$\frac{5}{120h}$	$\frac{5}{12h}$	0	$-\frac{5}{12h}$	$-\frac{5}{120h}$	0
$B''_{5,i}(x)$	0	$\frac{1}{6h^2}$	$\frac{1}{3h^2}$	$-\frac{1}{h^2}$	$\frac{1}{3h^2}$	$\frac{1}{6h^2}$	0

Table (3.4) shows the values of $B_{5,i}(x)$'s and their derivatives.

3.4. B-Spline Interpolation

The set $\{B_{l,i}(x)\}_{i=-l}^{n-1}$ is defined as a basis for $S_l(\Omega_n)$, so any spline $s \in S_l(\Omega_n)$ can be written as:

$$s(x) = \sum_{i=-l}^{n-1} C_i B_{l,i}(x) \qquad (3.14)$$

Given a function $f:[a,b] \to R$, we can find $s \in S_l(\Omega_n)$, such that $s(x_j) = f(x_j)$, where $0 \le j \le n$. This interpolation problem has (l-1) free parameter, where (l) is the degree of the b-spline interpolation. In the next section we will describe the b-spline interpolation of the high degree started form the second degree b-spline interpolation.

3.4.1. The Second Degree B-spline Interpolation

In the second degree B-spline interpolation we have to built a system of order $(n+1)\times(n+2)$ equations.

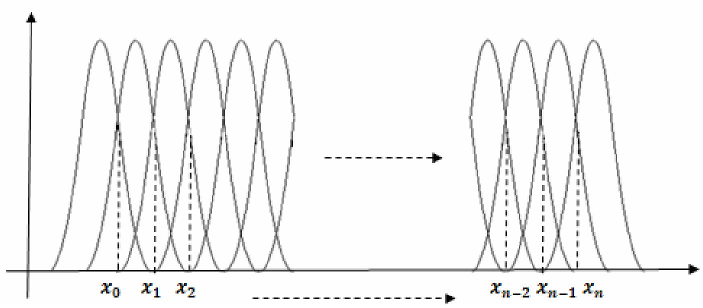

Fig. (3.10) shows $B_{2,i}(x)$ basis

This means we need to add one free parameter to solve this system because this system is not a square. For example, the first derivative at the x_0 is a good enough to complete this system.

Now, we can construct the whole system of the equations by using the general form (3.14) of the second degree b-spline interpolation:

First of all, at any nodes x_i where $i = 0,1,2,...,n$, we have

$$s(x_i) = C_{i-2}B_{2,i-2}(x_i) + C_{i-1}B_{2,i-1}(x_i) = f(x_i) \tag{3.15}$$

Where $i = 0,1,2,...,n$

We note that at the nodes we have only two C's are not zero and the other C's are zero because there is only two basses intersect at the nodes in the second degree b-spline, this is clear in figure (3.10). Then we have (n+1) nodes and (n+2) basis, this leads to $(n+1)\times(n+2)$ system of equations which is not square matrix. We should add one free parameter fore example the first derivative at x_0 is enough to do that;

$$s'(x_0) = C_{-2}B'_{2,-2}(x_0) + C_{-1}B'_{2,-1}(x_0) = f'(x_0) \tag{3.16}$$

Then the system is defined as the following form:

$$\begin{bmatrix} -\frac{1}{h} & \frac{1}{h} & 0 & . & . & . & . & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & . & . & . & . & . \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ . & . & . & . & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & . & . & . & . & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ . \\ . \\ . \\ . \\ . \\ . \\ C_{n-2} \\ C_{n-1} \end{bmatrix} = \begin{bmatrix} f'(x_0) \\ f(x_0) \\ f(x_1) \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ f(x_{n-2}) \\ f(x_{n-1}) \end{bmatrix} \tag{3.17}$$

Solving this system will give us the value of the C's. Then substitute it in the original formula (2.14) to found the best piecewise interpolation functions of the second degree b-spline interpolation.

3.4.1.1. Example

Consider the function

$$f(x)=\begin{cases}-(x-1)^2, & if \;\; x\in[0,1].\\ 0, & if \;\; x\in[1,2].\\ (x-2)^2, & if \;\; x\in[2,3].\end{cases} \tag{3.18}$$

Where $f'(0)=2,\ and\ h=1$.

Now we apply the system (3.17) to approximate this function by second degree b-spline. Then the system of the equations as the following form:

$$\begin{bmatrix}-2 & 2 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0\\ 0 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 1 & 1\end{bmatrix}\begin{bmatrix}C_{-2}\\ C_{-1}\\ C_0\\ C_1\\ C_2\end{bmatrix}=\begin{bmatrix}4\\ -2\\ 0\\ 0\\ 2\end{bmatrix}$$

Solving this system leads to $C_{-2}=-2, C_{-1}=0, C_0=0, C_1=0, and\ C_2=2$.

Now we have:

$$\mathrm{S(x)}=\sum_{\mathrm{i=-2}}^{\mathrm{n-1}}\mathrm{C_i}\, B_{2,i}(x)=C_{-2}B_{2,-2}(x)+C_2B_{2,2}(x). \tag{3.19}$$

But $B_{2,-2}(x)=\dfrac{(1-x)^2}{2}$ and $B_{2,2}(x)=\dfrac{(x-2)^2}{2}$,

Then

$$S(x)=\begin{cases}-(1-x)^2, & if \;\; x\in[0,1].\\ (x-2)^2, & if \;\; x\in[2,3].\\ 0, & otherwise.\end{cases}$$

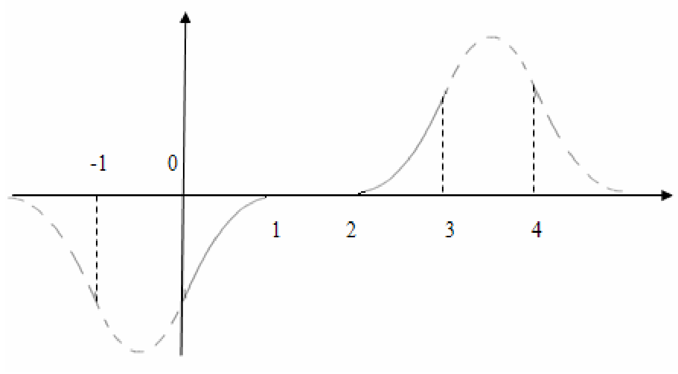

Fig. (3.11) shows the solution of example (2.3.1.1)

This result is clear in Figure (3.11) which is completely the same of our original function.

3.4.2. The Third Degree B-Spline Interpolation

In the third degree B-spline interpolation the system of the equations is more complicated than the second degree b-spline interpolation, because we have to built a system of order $(n+1)\times(n+3)$ equations, this means we need to add two free parameter to solve this system. These kinds of problems are called the boundary conditions problems or clamped problems, for example we can chose $f'(x_0)$ and $f'(x_n)$, *or* $f'(x_0)$ *and* $f''(x_0)$ to complete and solve this system. In figure (3.12), we can denote this kind of structure.

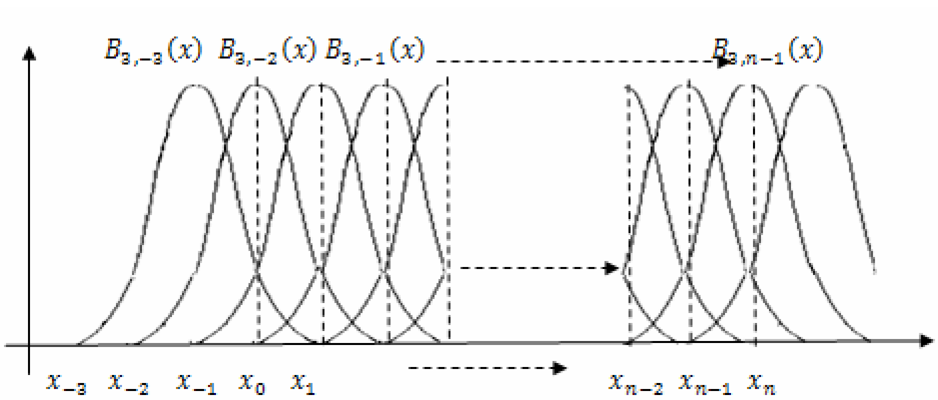

Fig. (3.12) shows $B_{3,i}(x)$

In the figure (2.12), we can denote that at any x_i where i = 0, 1, 2, 3, … , n there are three basis ($B_{3,i}(x)$) not equal to zero. For example, at x_0, $B_{3,-3}(x_0)$, $B_{3,-2}(x_0)$, and $B_{3,-1}(x_0)$ is not equal to zero, this means by equation (2.14) we have:

$$s(x_i) = C_{i-3}B_{3,i-3}(x_i) + C_{i-2}B_{3,i-2}(x_i) + C_{i-1}B_{3,i-1}(x_i) = f(x_i) \tag{3.20}$$

Where i = 0, 1, 2, …, n.·

Now, the boundary conditions or clamped conditions will be added to our system of linear equations to solve it. For example, the clamped condition is enough to complete our system which was taken from (2.16).

Then

$$S'(x_0) = C_{-3}B'_{3,-3}(x_0) + C_{-2}B'_{3,-2}(x_0) + C_{-1}B'_{3,-1}(x_0) = f'(x_0) \tag{3.21}$$

$$S'(x_n) = C_{n-3}B'_{3,n-3}(x_n) + C_{n-2}B'_{3,n-2}(x_n) + C_{n-1}B'_{3,n-1}(x_n) = f'(x_n) \tag{3.22}$$

Now, by equations (3.20), (3.21), and (3.22) the following system will be constructing at all the nodes belonging to the interval [a, b] as this form:

$$\begin{bmatrix} -\frac{3}{h} & 0 & \frac{3}{h} & 0 & 0 & . & . & . & 0 \\ 1 & 4 & 1 & 0 & 0 & . & . & . & . \\ 0 & 1 & 4 & 1 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & 1 & 4 & 1 & 0 \\ . & . & . & . & . & 0 & 1 & 4 & 1 \\ 0 & . & . & . & . & 0 & -\frac{3}{h} & 0 & \frac{3}{h} \end{bmatrix} \begin{bmatrix} C_{-3} \\ C_{-2} \\ C_{-1} \\ C_0 \\ . \\ . \\ . \\ . \\ C_{n-2} \\ C_{n-1} \end{bmatrix} = 6 \begin{bmatrix} f'(x_0) \\ f(x_0) \\ f(x_1) \\ . \\ . \\ . \\ . \\ . \\ f(x_n) \\ f'(x_n) \end{bmatrix} \tag{3.23}$$

The first and the last rows applied the conditions (3.21) and (3.22) respectively. After solving the system (3.23), we can find the C_i where i = -3, -2, -1, 0, … , n-1, and then substitute it in the equation (3.14) to approximate the original function. The following example will be illustrating the previous steps practically which is very important to understanding the idea.

3.4.2.1. Example

Let $f(x)=\cos(\pi x)$ is defined on the interval [a, b] = [0, 1], with h = 1/2, then we have $\Omega=\left\{0, \frac{1}{2}, 1\right\}$, we approximate this function by cubic b-spline which was illustrated in the previous section by clamped conditions $f'(0)=f'(1)=0$, this means, we have to find $S(x)$ satisfies these conditions, then

$$S(x)=\sum_{i=-3}^{n-1} C_i B_{3,i}(x)=\sum_{i=-3}^{1} C_i B_{3,i}(x) \tag{3.24}$$

with boundary conditions at x_0 we have $S'(x_0)=f'(x_0)$, and at x_n we have $S'(x_n)=f'(x_n)$, then

$$S'(x_0)=\sum_{i=-3}^{0} C_i B'_{3,i}(x_0)$$

$$=C_{-3}B'_{3,-3}(0)+C_{-2}B'_{3,-2}(0)+C_{-1}B'_{3,-1}(0)+C_0B'_{3,0}(0)+C_1B'_{3,1}(0)=0 \tag{3.25}$$

Use the derivative of the nodes in table (2.2) to get:

$$C_{-3}\left(-\frac{1}{2h}\right)+C_{-2}(0)+C_{-1}\left(\frac{1}{2h}\right)=0 \quad \Rightarrow \quad -C_{-3}+C_{-1}=0 \tag{3.26}$$

And by the same way at x_n we have:

$$S'(x_n)=S'(1)$$

$$=C_{-3}B'_{3,-3}(1)+C_{-2}B'_{3,-2}(1)+C_{-1}B'_{3,-1}(1)+C_0B'_{3,0}(1)+C_1B'_{3,1}(1)=0 \tag{3.27}$$

This means we get the second equation:

$$C_{-1}\left(-\frac{1}{2h}\right)+C_0(0)+C_1\left(\frac{1}{2h}\right)=0 \quad \Rightarrow \quad -C_{-1}+C_1=0 \tag{3.28}$$

Now the system of equation (3.23) leads to the form:

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_{-3} \\ C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 0 \\ -6 \\ 0 \end{bmatrix} \qquad (3.29)$$

Solving the equation (3.29) will give us the value of C_i's whish are:

$$C_{-3} = 0, C_{-2} = 1.5, C_{-1} = 0, C_0 = -1.5, C_1 = 0$$

Now, substitute these values of C_i's in the formal equation (3.14) to find the approximation interpolation by cubic b-spline which gives us the following two results, because we divided our interval [0, 1] to two subintervals which are [0, 0.5], and [0.5, 1]. That means we have two equations. The first one is:

$$S_1(x) = C_{-3}B_{3,-3}(x) + C_{-2}B_{3,-2}(x) + C_{-1}B_{3,-1}(x) + C_0B_{3,0}(x) \qquad (3.30)$$

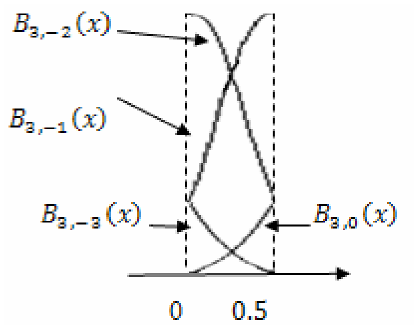

Fig.(3.13) shows the $B_{3,i}(x)$

We take the only these $B_{3,i}(x)$'s, where i = -3, -2, -1, and 0. Because it through in the interval [0, .5], which is the others are not through, this is clear in figure (3.13).

Then, by collect these functions the first approximation function will be come out as:

$$S_1(x) = 4x^3 - 6x^2 + 1$$

By the same way we can found the second one, which is:

$$S_2(x) = C_{-2}B_{3,-2}(x) + C_{-1}B_{3,-1}(x) + C_0B_{3,0}(x) + C_1B_{3,1}(x) \qquad (3.31)$$

Then $S_2(x) = 4x^3 - 6x^2 + 1$

The maximum error is:

$Maxerror = \|S_i(x) - f(x)\| = 0.02$, where i=1, and 2

Note:

This error is looks like big, because we divided the interval [0, 1] to two subintervals only. If the interval is divided to many subintervals then the errors will be decrease to small and small. And there is anther notes, which is the outside of our interval [0, 1] has a big error because we interst with the interior nodes only. See figure(3.14).

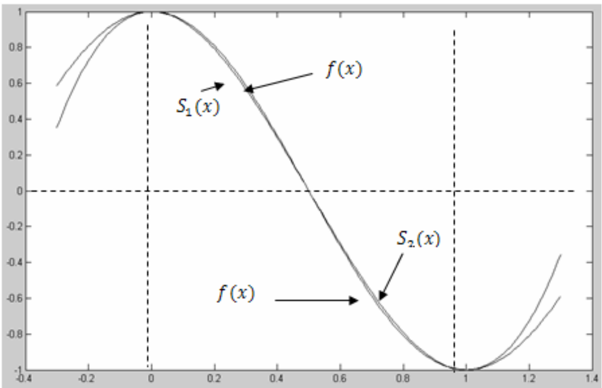

Fig. (3.14) shows the original function $f(x)$, and the b-spline interpolation function $S_i(x)$

3.4.3. The Fourth Degree B-spline Interpolation

In this part we will try to brevity, because the fourth b-spline has the same as the way to built it as the cubic b-spline interpolation. The important changing deference than the cubic b-spline is tries to find and add a new condition to solve their system of equations.

The order of the system of the fourth b-spline interpolation is $(n+1)\times(n+4)$ equations which are not the same as the third degree b-spline. This means the three boundary conditions are needed to solve this system. As an example, we will take the follows clamed conditions:

$S'(x_0)=f'(x_0), S'(x_n)=f'(x_n)$, and $S''(x_0)=f''(x_0)$, or $S''(x_n)=f''(x_n)$. (3.32)

If we denote the coefficient matrix by A, this system of equations will be generating the following matrix:

$$A=\begin{bmatrix} \frac{1}{2h^2} & -\frac{1}{2h^2} & -\frac{1}{2h^2} & \frac{1}{2h^2} & 0 & 0 & . & . & . & 0 \\ -\frac{1}{6h} & -\frac{1}{6h} & \frac{1}{6h} & \frac{1}{6h} & 0 & 0 & . & . & . & . \\ \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} & 0 & 0 & . & . & . & . \\ 0 & \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} & 0 & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} & 0 \\ . & . & . & . & . & 0 & \frac{1}{24} & \frac{11}{24} & \frac{11}{24} & \frac{1}{24} \\ 0 & . & . & . & . & . & -\frac{1}{6h} & -\frac{1}{6h} & \frac{1}{6h} & \frac{1}{6h} \end{bmatrix} \quad (3.33)$$

It's clear that, the first, the second, and the last row are defined the clamped conditions.

And the reminder of this system $A \cdot C = B$ can be defined as C and B respectively, which are of the forms:

$$C=\begin{bmatrix} C_{-4} \\ C_{-3} \\ C_{-2} \\ . \\ . \\ . \\ . \\ . \\ . \\ C_{n-2} \\ C_{n-1} \end{bmatrix}, \textit{and}\ B=\begin{bmatrix} f''(x_0) \\ f'(x_0) \\ f(x_0) \\ f(x_1) \\ . \\ . \\ . \\ . \\ . \\ f(x_n) \\ f'(x_n) \end{bmatrix} \quad (3.34)$$

Finally, after solving this system the value of C's are found by $C = A^{-1} \cdot B$, and substitute it in the equation (3.14) to find the fourth degree piecewise b-spline interpolation functions. In the figure (3.15) we can denote that at the interval $[x_i, x_{i+1}]$, there are five equations not equal to zero.

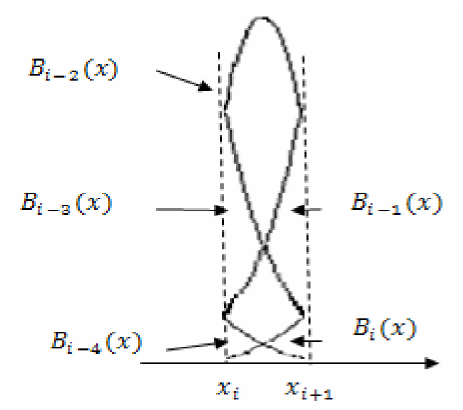

Fig. (3.15) shows $B_{4,i}(x)$ basis in $[x_i, x_{i+1}]$.

This means at any piecewise subinterval the following form of b-spline interpolation is appear:

$$S_i(x) = \sum_{j=i-4}^{i} C_j B_{4,j}(x), \text{ where i = 0, 1, 2, 3, . . ., n-1.} \qquad (3.35)$$

3.4.4. The Fifth Degree B-spline Interpolation

By the same way in the previous sections, the fifth degree b-spline interpolation can be built. But now we need to add anther condition to solve this system of the linear equations of fifth degree b-spline interpolation because this system has order of $(n+1)\times(n+5)$, which is not a square system. See figure (3.16).

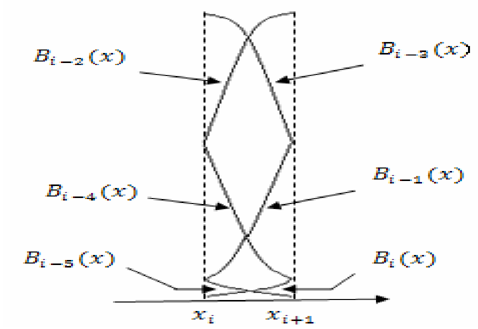

Fig. (3.16) shows $B_{5,i}(x)$ basis in $[x_i, x_{i+1}]$

For example, the clamped conditions are necessary to add to this system. We will take these conditions as the follows form:

$$S'(x_0)=f'(x_0),\ S'(x_n)=f'(x_n),\ S''(x_0)=f''(x_0),\ and\ S''(x_n)=f''(x_n) \qquad (3.36)$$

If we defined again the coefficient matrix by A, this system of the linear fifth degree b-spline interpolation is defined as $A\cdot C=B$, where A, C, and B as the follows forms

$$A=\begin{bmatrix} \frac{1}{6h^2} & \frac{1}{3h^2} & \frac{-1}{h^2} & \frac{1}{3h^2} & \frac{1}{6h^2} & 0 & . & . & . & 0 \\ \frac{-5}{120h} & \frac{-5}{12h} & 0 & \frac{5}{12h} & \frac{5}{120h} & 0 & . & . & . & . \\ \frac{1}{120} & \frac{26}{120} & \frac{66}{120} & \frac{26}{120} & \frac{1}{120} & 0 & . & . & . & . \\ 0 & \frac{1}{120} & \frac{26}{120} & \frac{66}{120} & \frac{26}{120} & \frac{1}{120} & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & 0 & \frac{1}{120} & \frac{26}{120} & \frac{66}{120} & \frac{26}{120} & \frac{1}{120} & 0 \\ . & . & . & . & 0 & \frac{1}{120} & \frac{26}{120} & \frac{66}{120} & \frac{26}{120} & \frac{1}{120} \\ . & . & . & . & 0 & \frac{-5}{120h} & \frac{-5}{12h} & 0 & \frac{5}{12h} & \frac{5}{120h} \\ 0 & . & . & . & 0 & \frac{1}{6h^2} & \frac{1}{3h^2} & \frac{-1}{h^2} & \frac{1}{3h^2} & \frac{1}{6h^2} \end{bmatrix} \qquad (3.37)$$

$$C=\begin{bmatrix} C_{-5} \\ C_{-4} \\ C_{-3} \\ . \\ . \\ . \\ . \\ . \\ . \\ C_{n-2} \\ C_{n-1} \end{bmatrix},\ and\ B=\begin{bmatrix} f''(x_0) \\ f'(x_0) \\ f(x_0) \\ f(x_1) \\ . \\ . \\ . \\ . \\ f(x_n) \\ f'(x_n) \\ f''(x_n) \end{bmatrix} \qquad (3.38)$$

After solving this system the C's are found by $C = A^{-1} \cdot B$, and we substitute it in (3.14) to approximate the original function by fifth degree b-spline interpolation piecewise functions.

3.5. Case Study

B-Spline interpolation compared with finite difference, finite elements and finite volume methods which applied to two-point boundary value problems.

3.5.1. Introduction

After the b-spline interpolation section have introduced, it is important to apply this method to solve some problems which have not solved exactly yet. This section considers the comparing study between cubic b-spline interpolation and the subject that was published in [14] to solve the two-point boundary value problems as the form:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) = f(x),\ a < x < b \tag{3.39}$$

$$u(a) = u(b) = 0, where\ p \in C^1[a,b]. \tag{3.40}$$

3.5.2. Problem

Now, we will take $p(x) = e^{1-x}$ and $f(x) = 1 + e^{1-x}$ in the interval $[a,b] = [0,1]$ as an applied example see [8] to see how dose the cubic b-spline interpolations is the best method than the other methods?

The analytic solution for this problem is $u(x) = x\left(1 - e^{x-1}\right)$. We obtain from (3.39) the equation of this form:

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) = f(x) \Rightarrow -\frac{d}{dx}\left(e^{1-x}\frac{du}{dx}\right) = 1 + e^{1-x}$$

Then

$$\frac{d^2u}{dx^2} - \frac{du}{dx} = -e^{x-1} - 1 \tag{3.41}$$

The b-spline solution can be obtained by using (3.41) in (3.24) for n = 10 and $l = 3$, because the third degree b-spline will be used to solve this problem, and the interval will be divided to ten subintervals.

Now, we apply the general form of the cubic b-spline interpolation and their derivatives on the defined interval, then

$$u(x) = C_{-3}B_{-3}(x) + C_{-2}B_{-2}(x) + \dots + C_{-9}B_{-9}(x),$$

$$u'(x) = C_{-3}B'_{-3}(x) + C_{-2}B'_{-2}(x) + \dots + C_{-9}B'_{-9}(x),$$

And $$u''(x) = C_{-3}B''_{-3}(x) + C_{-2}B''_{-2}(x) + \dots + C_{-9}B''_{-9}(x) \qquad (3.42)$$

We have the first derivative at nodes are:

$$u'(0) = C_{-3}\left(\frac{-1}{2h}\right) + C_{-2}(0) + C_{-1}\left(\frac{1}{2h}\right),$$

$$u'(0.1) = C_{-2}\left(\frac{-1}{2h}\right) + C_{-1}(0) + C_{0}\left(\frac{1}{2h}\right),$$

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$$u'(1) = C_{7}\left(\frac{-1}{2h}\right) + C_{8}(0) + C_{9}\left(\frac{1}{2h}\right). \qquad (3.43)$$

And also have the second derivatives at nodes are:

$$u''(0) = C_{-3}\left(\frac{1}{2h^2}\right) + C_{-2}\left(\frac{-2}{h^2}\right) + C_{-1}\left(\frac{1}{2h^2}\right),$$

$$u''(0.1) = C_{-2}\left(\frac{1}{2h^2}\right) + C_{-1}\left(\frac{-2}{h^2}\right) + C_{0}\left(\frac{1}{2h^2}\right),$$

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$$u''(1) = C_7\left(\frac{1}{2h^2}\right) + C_8\left(\frac{-2}{h^2}\right) + C_9\left(\frac{1}{2h^2}\right) \qquad (3.44)$$

Now, substitute the equations (3.43), and (3.44) in the general problem in equation (3.41). Then the result of equations can be written in a matrix form such that:

$$A = \begin{bmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & . & . & . & 0 \\ 2+h & -4 & 2-h & 0 & . & . & . & . \\ 0 & 2+h & -4 & 2-h & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & 0 & 2+h & -4 & 2-h & 0 \\ . & . & . & . & 0 & 2+h & -4 & 2-h \\ 0 & . & . & . & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{bmatrix} \qquad (3.45)$$

Add he first row and the last row as the clamped conditions. They come from equation $u(0) = 0, and\ u(1) = 0$. The order of the matrix A is (13×13), and for C and B defined as the forms:

$$C = \begin{bmatrix} C_{-3} \\ C_{-2} \\ . \\ . \\ . \\ C_9 \end{bmatrix}, \ and\ B = 2h^2 \begin{bmatrix} -1.3679 \\ -1.4066 \\ . \\ . \\ . \\ -2 \end{bmatrix} \qquad (3.46)$$

The previous of the linear system $A \cdot C = B$ can be solved to find the values of C's,

For example (5.1) in [14], $h = 0.1$ have been taken, then the values of C's will be find respectively:

$$C = [-0.0657 \quad 0.0012 \quad 0.0608 \quad 0.1119 \quad 0.1531 \ \ldots \ 0.0050 \quad -0.1102]$$

Then the approximation solution of the exact solution can be found by the cubic b-spline interpolation as follows:

$$S(x) = \sum_{i=-3}^{n-1} C_i B_{3,i}(x) \qquad (3.47)$$

Then

$S_1(x) = -0.2000x^3 - 0.3649x^2 + 0.6325x - 0.000016$, at the interval $[0, 0.1]$,

$S_2(x) = -0.2333x^3 - 0.3550x^2 + 0.6315x + 0.000016$, at the interval $[0.1, 0.2]$

$S_3(x) = -0.2500x^3 - 0.3449x^2 + 0.6294x + 0.000015$, at the interval $[0.2, 0.3]$,

$S_4(x) = -0.3000x^3 - 0.2999x^2 + 0.6159x + 0.0014$, at the interval $[0.3, 0.4]$,

$S_5(x) = -0.3166x^3 - 0.2800x^2 + 0.6080x + 0.00526$, at the interval $[0.4, 0.5]$,

$S_6(x) = -0.400x^3 - 0.1549x^2 + 0.5455x + 0.012983$, at the interval $[0.5, 0.6]$,

$S_7(x) = -0.4166x^3 - 0.1250x^2 + 0.5275x + 0.01658$, at the interval $[0.6, 0.7]$,

$S_8(x) = -0.4666x^3 - 0.01999x^2 + 0.4539x + 0.03373$, at the interval $[0.7, 0.8]$,

$S_9(x) = -0.6000x^3 + 0.3000x^2 + 0.1979x + 0.1020$, at the interval $[0.8, 0.9]$,

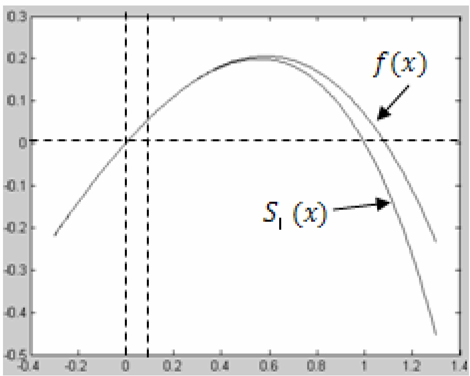
$S_{10}(x) = -0.6000x^3 + 0.3000x^2 + 0.1979x + 0.1020$, at the interval $[0.9, 1]$,

For more detail see [8]. If we compare our results with the results published in [14], we will see the difference between them and conclude that the B-spline interpolation is the better methods to interpolate any smooth functions than others. The numerical results for the previous example are shown in Table (3.5), which shows that there are big differences errors between B-spline interpolation and the other methods unless there is no remarkable difference among the accuracy of the other three methods in the case where f is sufficiently smooth.

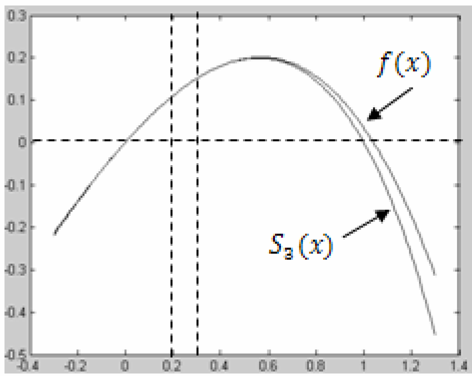
Methods	h	Max-norm/h^2
Finite difference solution	0.1 0.01	8.24 e-3 8.31e-3
Finite element solution	0.1 0.01	6.35 e-3 6.36 e-3
Finite volume solution	0.1 0.01	3.18 e-3 3.18 e-3
B-spline interpolation	0.1 0.01	2.9 e-4 2.8 e-6

Table (3.5) shows the comparing result of b-spline method and other method

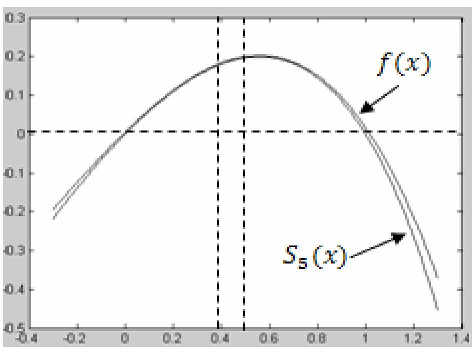
The figure (3.17) will be shows these results of some subintervals of $[a,b]$.



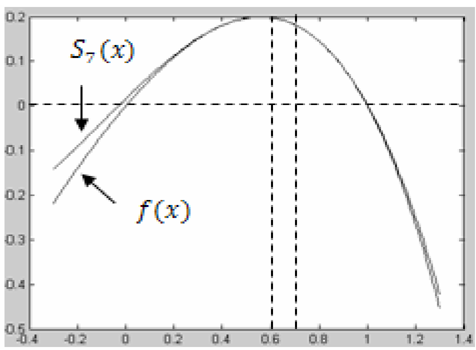
$S_1(x)$ at the interval [0, 0.1]



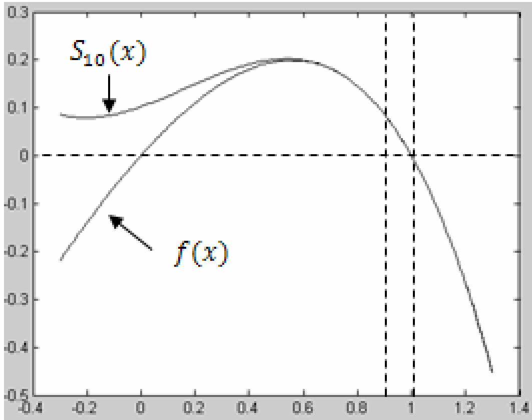
$S_3(x)$ at the interval [0.2, 0.3]



$S_5(x)$ at the interval [0.4, 0.5].



$S_7(x)$ at the interval [0.6, 0.7]



$S_{10}(x)$ at the interval [0.9, 1].

Fig(3.17) shows the graph of the piecewise b-spline interpolation in problem (3.2) and the exact solution of $f(x)$

4. Perturbation Theory

Mathematically, Perturbation theory comprises mathematical methods that are used to find an approximate solution to a problem that cannot be solved exactly, by starting from the exact solution of a related problem.

The general two-point boundary-value problems involve a second-order differential equation of the form:

$$y'' = f(x, y, y'), \; a \le x \le b, \qquad (4.1)$$

Together with the boundary conditions:

$$y(a) = \alpha, \; and \;\; y(b) = \beta \qquad (4.2)$$

Most of the material concerning second-order boundary-value problems can be extended to problems with boundary conditions of the form:

$$\alpha_1 y(a) - \beta_1 y'(a) = \alpha, \; and \;\; \alpha_2 y(a) - \beta_2 y'(b) = \beta \qquad (4.3)$$

Where $|\alpha_1| + |\beta_1| \ne 0$, and, $|\alpha_2| + |\beta_2| \neq 0$ but some of the techniques become quite complicated.

4.1. Introduction

The approximate solution of boundary-value problems with a small parameter affecting highest derivative of the differential equation is described. It is a well-known fact that the solution of singularly perturbed boundary-value problem exhibits a multiscale character. That is, there is a thin layer where the solution varies rapidly, while away from the layer the solution behaves regularly and varies slowly. This class of problems has recently gained importance in the literature for two main reasons. Firstly, they occur frequently in many areas of science and engineering, for example, combustion, chemical reactor theory, nuclear engineering, control theory, elasticity, Euid mechanics etc. A few notable examples are boundary-layer problems, WKB Theory, the modeling of steady and unsteady viscous Eow problems with large Reynolds number and convective heat transport problems with large Peclet number. Secondly, the occurrence of sharp boundary-layers as ε, the coefficient of highest derivative, approaches zero creates difficulty for most standard numerical schemes. There exist a variety of techniques for solving singularly perturbed boundary value problems. The numerical solution of two point boundary-value problems using high degree b-spline has been considered.

4.2. Derivation of the method

The general form of singularly perturbed boundary value problem is defined as:

$$\varepsilon y'' + p(x)y' + q(x)y = r(x) \qquad (4.4)$$

Such that $p(x)$, $q(x)$, *and* $r(x)$ are smooth and bounded functions. It is known that problem exhibits boundary layers at one or both ends of the interval depending on the properties of $p(x)$ see [11]. This kind of problems occur in many engineering fields especially that interested in applied mathematics such as quantum mechanics, chemical reactor theory, optimal control, reaction-diffusion process ect.

Before starting, we have to introduce the basic concepts of this study. First of all, the meaning of perturbed theory comprises the mathematical methods that are used to find an approximate solution to a problem which cannot be solved exactly, Perturbation theory is applicable if the problem can be formulated by adding a ”small” term to the mathematical description of the exactly solvable problem.

Second, the shape preserving means that the interpolated polynomial has the same property as the original function as. These properties are the smoothness and the increasing or the decreasing of our function.

In fact, a lot of people think that if the first derivative terms are not given at the boundary value problems then the high B-spline interpolation can't be used to solve the singularly perturbed boundary value problems because the high b-spline interpolation needs more and more conditions depends on their degrees. This study utilizes the new scheme that approximate the first derivative at the boundary values by using the second degree polynomial that passes the nodes $(x_i, f(x_i))$ see [13].

4.3. The brief of the used scheme

If we have S as polynomial spline of the second degree, and we have a free parameter $d = S'(a)$, where $\Omega_n = \{x_0, x_1, x_2, ..., x_n\}$ is a set of nodes of a partition on the interval [a, b], such that $a = x_0 < x_1 < x_2 < ... < x_n = b$, and $f(x_i) = y_i$ is given for all $i = 0, 1, 2, ..., n$, and suppose $S_2(x)$ is continuous on [a, b] and differentiable that is $S_2 \in C^1[a,b]$, then we can find $S'(a) = S'(x_0) = d$ such that $S \in S_2(\Omega_n, f)$ is shape preserving. For more detail of this scheme; denote S_i as the second-degree polynomial on the interval $I_i = [x_i, x_{i+1}]$ as a restriction of S such that $S_i = S|_{I_i}$, also we assume $1,\ (x - x_i),\ (x - x_i)^2$ as a basis, this means that at any nodes x_i where i = 0, 1, 2, …,n, we get:

$$S_i(x) = S(x_i) + S'(x_i)(x - x_i) + \left[\frac{S[x_{i+1}, x_i] - S'(x_i)}{x_{i+1} - x_i}\right](x - x_i)^2 \tag{4.5}$$

Collect and put all of the given information in a matrix forms denoted by q, in this matrix the hypotheses will be decreeing as mush as possible until the following form will be appear:

$$q_1 = 0, \textit{ and } q_{i+1} = 2\left[\frac{y_{i+1} - y_i}{x_{i+1} - x_i}\right] - q_i \tag{4.6}$$

Then we have to decide that whether the function is monotony decreasing or increasing, if denote d as the interval which contains the best of the derivative then:

$$d = \begin{cases} max\,\{q_i : i = 2k\} \le d \le -max\,\{q_i : i = 2k-1\} \\ \quad where\;\; k = 1,2,3,...,n, if\;\; S\; is\;\; monoton\, dec. \\ -min_{i\; add}(q_i) \le d \le min_{i\; even}(q_i)\;\; if\;\;\; S\; is\; monoton\; inc. \end{cases} \tag{4.7}$$

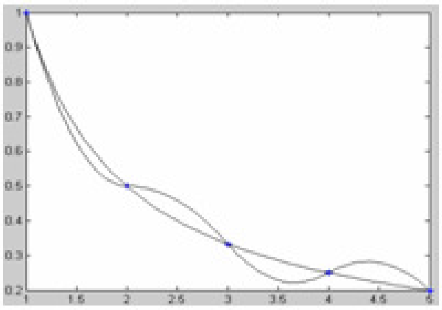
Now, a matlab program can be written to solve this problem (4.6) by employing the equation (4.7) to determine the smallest interval which contains the best derivative at the boundary points. For more detail see [13].The next example will be apply this scheme to understanding these steps.

4.3.1. Example

Let $f(x) = \dfrac{1}{x}$, *and* x = [1, 5], by the previous steps we can find the interval which contains the best derivatives a $x_0 = 1$, and compare our result by the exact derivative at $x_0 = 1$ which is $f'(1) = -1$. By a matlab [16] we can write a program of the previous steps to find the interval which contain the best derivative. This interval is $[-0.7726, -0.7725]$.

Now, we compute this result by a program (spl2), see appendix of programs, which is found and drawn the best interpolating spline polynomial of quadratic degree that passes the known nodes and uses the first derivative at x_0, by interring the two values $f'(1) = -1$. which is the exact derivative and $f'(1) = -0.7725$ which is the value that compute by previous step. The figure (3.1) and (3.2) are showing these results.

The errors are error1 = 0.0583 and error2 = 0.0239 showing in the Figure (4.1) and Figure (4.2) respectively. In the graph the result is clear, that means our scheme gives the best result. By the same way we can find $S'(x_n) = S'(b)$. Then, the high B-spline can be applied now to approximate the solution of the ‘‘Shape Preserving Linear Singularly Perturbed Boundary Value Problems”.



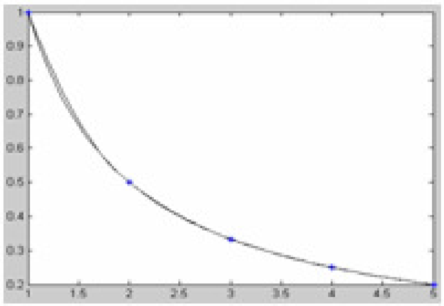


Fig. (4.1) shows I.S.P. by $f'(1) = -1$. **Fig. (4.2)** shows I.S.P. by $f'(1) = -0.7725$

4.4. Description of this method

After approximate the values of the first derivative at the boundary values of the main interval [a, b] which are $S'(a)$ and $S'(b)$ the forth and the fifth degree b-spline interpolation can be applied to find the solution of the shape preserving singularity perturbed boundary value problems by the following steps. First of all, the problem will be solved by cubic B-spline method to find the best of the approximation nodes $(x_i, f(x_i))$ by using the previous section. The general form of the linear singularly perturbed boundary value problem as the form:

$$\varepsilon y'' + p(x)y' + q(x)y = r(x), \tag{4.8}$$

where $y(a) = \alpha,\ y(b) = \beta,\ and\ 0 < \varepsilon < 1.$

The cubic b-spline basis will be used to solve this problem which is:

$$B_{3,i}(x) = \frac{1}{6h^3}\begin{cases} (x - x_i)^3, & if\ \ x \in [x_i, x_{i+1}]. \\ h^3 + 3h^2(x - x_{i+1}) + 3h(x - x_{i+1})^2 \ldots & \\ \qquad -3(x - x_{i+1})^3, & if\ \ x \in [x_{i+1}, x_{i+2}]. \\ h^3 + 3h^2(x_{i+3} - x) + 3h(x_{i+3} - x)^2 \ldots & \\ \qquad -3(x_{i+3} - x)^3, & if\ \ x \in [x_{i+3}, x_{i+4}]. \\ (x_{i+4} - x)^3 & if\ \ x \in [x_{i+4}, x_{i+5}]. \\ 0, & otherwise. \end{cases} \tag{4.9}$$

Then construct the following system by the previous basis in the general form (4.8) of the linear singularly perturbed boundary value problem which is of the form:

$$\begin{bmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 & . & . & . & 0 \\ u_0 & v_0 & s_0 & 0 & . & . & . & . \\ 0 & u_1 & v_1 & s_1 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & 0 & u_i & v_i & s_i & 0 & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & 0 & u_n & v_n & s_n \\ 0 & . & . & . & 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} C_{-3} \\ C_{-2} \\ C_{-1} \\ . \\ . \\ . \\ . \\ C_{n-2} \\ C_{n-1} \end{bmatrix} = \begin{bmatrix} \alpha \\ r(x_0) \\ r(x_1) \\ . \\ . \\ . \\ . \\ r(x_n) \\ \beta \end{bmatrix} \tag{4.10}$$

Where $u_i = \frac{\varepsilon}{h^2} - \frac{p(x_i)}{2h} + \frac{q(x_i)}{6}, v_i = \frac{-2\varepsilon}{h^2} + \frac{2q(x_i)}{3}, \text{ and } s_i = \frac{\varepsilon}{h^2} + \frac{p(x_i)}{2h} + q(x_i)$

For i = 0, 1, 2, 3, …, n.

Also we have the first and the last raw are the boundary conditions:

$$y(a) = \alpha, \text{ and } y(b) = \beta. \tag{4.11}$$

This system as the form $A \cdot C = B$, then solve this system to find C, and substitute it in the cubic B-spline interpolation form:

$$S_f(x) = \sum_{i=-3}^{n-1} C_i B_{3,i}(x) \tag{4.12}$$

Then the $f(x_i)$ where i = 0,1, 2, 3, …, n-1, can be approximated from (4.12).

Second, the scheme which is in the previous section of finding the best derivative at the boundary value can be applied now, because all the values of nodes are approximated, Then it's known, the next step gives the approximation of $y'(a) \cong \delta$, and $y'(b) \cong \gamma$. This means we have a new information to apply the forth degree b-spline interpolation method if we use $y'(a) \cong \delta$ only or the fifth degree B-spline interpolation method if we use the $y'(b) \cong \gamma$ too. It's interested if we apply the cubic B-spline interpolation to solve the shape preserving singularity perturbed boundary value problems; this means without use the first derivative at the boundary values. We will do this and compare the error of the cubic b-spline interpolation results with the error of the fifth degree B-spline interpolation results later. The fifth degree B-spline as follows:

$$B_{5,i}(x)=\frac{1}{120h^5}\begin{cases}(x-x_i)^5, & if\ \ x\in[x_i,x_{i+1}].\\ h^5+5h^4(x-x_{i+1})+10h^3(x-x_{i+1})^2+10h^2(x-x_{i+1})^3\ldots & \\ \quad +5h(x-x_{i+1})^4-5(x-x_{i+1})^5, & if\ \ x\in[x_{i+1},x_{i+2}].\\ 26h^5+50h^4(x-x_{i+2})+20h^3(x-x_{i+2})^2-20h^3(x-x_{i+2})^3\ldots & \\ \quad -20h(x-x_{i+2})^4+10(x-x_{i+2})^5, & if\ \ x\in[x_{i+2},x_{i+3}].\\ 26h^5+50h^4(x_{i+4}-x)+20h^3(x_{i+4}-x)^2-20h^3(x_{i+4}-x)^3\ldots & \\ -20h(x_{i+4}-x)^4+10(x_{i+4}-x)^5, & if\ \ x\in[x_{i+3},x_{i+4}].\\ h^5+5h^4(x_{i+5}-x)+10h^3(x_{i+5}-x)^2+10h^2(x_{i+5}-x)^3\ldots & \\ \quad +5h(x_{i+5}-x)^4-5(x_{i+5}-x)^5, & if\ \ x\in[x_{i+4},x_{i+5}].\\ (x_{i+6}-x)^5, & if\ \ x\in[x_{i+5},x_{i+6}].\\ 0, & otherwise.\end{cases} \quad (4.13)$$

Now, after all this information has been found the fifth degree B-spline can be applied by using the previous basis.

Finally, construct the following system by substitute the previous basis (4.13) in the general form (4.4) of the linear singularly perturbed boundary value problem, Then the system of the fifth degree B-spline will be appear like this:

$$A=\begin{bmatrix}\frac{-5}{120h} & \frac{-5}{12h} & 0 & \frac{5}{12h} & \frac{5}{120h} & 0 & . & . & \cdots & 0\\ \frac{1}{120} & \frac{26}{120} & \frac{66}{120} & \frac{26}{120} & \frac{1}{120} & 0 & . & . & \cdots & .\\ u_0 & v_0 & s_0 & w_0 & z_0 & 0 & . & . & \cdots & .\\ 0 & u_1 & v_1 & s_1 & w_1 & z_1 & 0 & . & \cdots & .\\ . & . & . & . & . & . & . & . & \cdots & .\\ . & 0 & u_i & v_i & s_i & w_i & z_i & 0 & \cdots & .\\ . & . & . & . & . & . & . & . & \cdots & .\\ . & \cdots & . & . & 0 & u_n & v_n & s_n & w_n & z_n\\ . & \cdots & . & . & 0 & \frac{1}{120} & \frac{26}{120} & \frac{66}{120} & \frac{26}{120} & \frac{1}{120}\\ 0 & \cdots & . & . & 0 & \frac{-5}{120h} & \frac{-5}{12h} & 0 & \frac{5}{12h} & \frac{5}{120h}\end{bmatrix} \quad (4.14)$$

Where $u_i = \frac{\varepsilon}{6h^2} - \frac{5p(x_i)}{120h} + \frac{q(x_i)}{120}$, $v_i = \frac{\varepsilon}{3h^2} - \frac{5p(x_i)}{12h} + \frac{26q(x_i)}{120}$,

$s_i = \frac{-\varepsilon}{h^2} + \frac{66q(x_i)}{120}$, $w_i = \frac{\varepsilon}{3h^2} + \frac{5p(x_i)}{12h} + \frac{26q(x_i)}{120}$, and $z_i = \frac{\varepsilon}{6h^2} + \frac{5p(x_i)}{120h} + \frac{q(x_i)}{120}$

i = 0, 1, 2, 3, …, n, and h = 1/n. (4.15)

For C and B defend as follows respectively.

$$C = \begin{bmatrix} C_{-5} \\ C_{-4} \\ C_{-3} \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ C_{n-1} \end{bmatrix}, \ and \ B = \begin{bmatrix} \delta \\ \alpha \\ r(x_0) \\ r(x_1) \\ . \\ . \\ . \\ . \\ r(x_n) \\ \beta \\ \gamma \end{bmatrix} \tag{4.16}$$

The first two rows and the last two rows are the boundary conditions. The system (4.14) and (4.16) lead to the form: $A \cdot C = B$, then compute C and substitute it in the following fifth degree B-spline interpolation formula to get the best result :

$$S_f(x) = \sum_{i=-5}^{n-1} C_i B_{5,i}(x) \tag{4.17}$$

This scheme will be demonstrated practically in the following example and compared with the results that published in [3].

4.4.1. Example

Consider the problem which is of the form of equation (4.1) as an example that published in [3] and [2] to apply the previous scheme, and the result will be comparing too.

$$\varepsilon y'' + y' + 2(1+x)y = \frac{1}{2}\exp\left(\frac{-x}{2}\right)\left[(1+x)(3-x) + \frac{\varepsilon}{2}\right] \tag{4.18}$$

With the boundary conditions:

$$y(0)=0,\ y(1)=\exp\left(\frac{-1}{2}\right)-\exp\left(\frac{-7}{3\varepsilon}\right) \quad (4.19)$$

The exact solution is:

$$y(x)=\exp\left(\frac{-x}{2}\right)-\exp\left[\frac{-x\left(x^{2}+3x+3\right)}{3\varepsilon}\right] \quad (4.20)$$

The solution of this example will be applying the previous steps, the result is demonstrated with the maximum absolute error: $max\left|y(x_i)-y_i\right|$ at the nodes. A deferent values of (ε) will be used where $h=\frac{1}{n}$. And also the differentiating of the boundary values by using the cubic B-spline depending on the values of (ε) will be found too. The next tables (4.1) and (4.2) will be shows all these results.

The maximum error table for n = 256:

ε	Kadalbajoo & Patidar's method	Aziz & Khan's method	Rajesh K. Bawa method
1/8	2.20e-04	3.934416e-05	1.007339e-09
1/16	1.10e-03	1.367532e-04	1.054689e-08
1/32	5.00e-03	5.116162e-04	1.429318e-07
1/64	2.30e-02	1.991524e-03	2.134858e-06
1/128	…	8.007187e-03	3.345245e-05

Table (4.1) shows the result of problem in [3] for n = 256.

Proposed method:

ε	Cubic B-spline	$\delta=f'(a)$	$\gamma=f'(b)$	Fifth degree B-spline
1/8	1.66118787e-05	7,499998	-0.30326887	9.32321797e-09
1/16	9.08945084e-05	15.50000	-.030327385	2.44180388e-08
1/32	4.42022286e-04	31.50000	-0.30327015	5.24520294e-08
1/64	1.80759128e-03	63.49999	-0.30327105	1.40487648e-06
1/128	7.62065557e-03	127.4998	-0.30322423	2.43253474e-05

Table (4.2) shows the present result for n = 256.

The maximum error table for n = 512:

ε	Kadalbajoo & Patidar's method	Aziz & Khan's method	Rajesh K. Bawa method
1/8	5.60e-05	9.835160e-06	6.295980e-11
1/16	2.70e-04	3.417675e-04	6.596570e-10
1/32	1.20e-03	1.277339e-04	8.926741e-09
1/64	5.50e-03	4.952798e-04	1.330488e-07
1/128	2.40e-02	1.959883-03	2.067402e-06

Table (4.3) shows the result of [3] for n = 512.

Proposed method:

ε	Cubic B-spline	$\delta = f'(a)$	$\gamma = f'(b)$	Fifth degree B-spline
1/8	4.16573269e-06	7,499998	-0.30326887	9.32321797e-09
1/16	2.27226995e-05	15.50000	-.030327385	2.44180388e-08
1/32	1.05016167e-04	31.50000	-0.30327015	5.24520294e-08
1/64	4.49877279e-04	63.49999	-0.30327105	1.40487648e-06
1/128	1.86820104e-03	127.4998	-0.30322423	2.43253474e-05

Table (4.4) shows the present result for n=512.

5. Conclusion

The boundary value problems constitute an ancient problem. There are a lot of science branches that relate in these kinds of the problems. Recently, there are numerous of scientists trying to solve these problems by different methods. In the first section two methods have been produced to solve it. In the second section, the b-spline definition and the five kinds of the basis have been described, at the end of the second section, which is in the case study, the boundary value problem has been solved by third degree b-spline interpolation and the result compared with the study which was published in [14]. That result concludes that the b-spline interpolation is the best way to give the best result. Not only was this, but also in the third section an alternative method was described to solve the linear singularly perturbed boundary value problems. The high degree b-spline was used to solve the perturbed boundary value problems. Again, the b-spline interpolation proved that it is the best way which is gives the best result, because the results were compared with the results that published in [2, 3]. Finally, throughout in the considering study concludes that the b-spline interpolation is one of the best methods to solve the boundary value problems until now. In the future, the third degree natural interpolation spline will be studied to approximate a better result.

References

[1] K. E. Atkinson, "An Introduction to Numerical Analysis", (1988). John Wiley & Sons, Inc. USA, QA297.A84.

[2] T. Aziz, A. Khan, "A spline method for second order singularly perturbed boundary-value problems, J. Comput. Appl. Math. 147 (2002) 445-452.

[3] R. K. Bawa. "Spline based computational technique for linear singularly perturbed boundary value problems". J. Compute. Appl. Math. 167 (2005) 225-236.

[4] S. C. Chapra, R. P. Canale. "Numerical Method for Engineers", 3-th edition, WCB/McGraw-Hill, (1998), TA345.C47.

[5] H. N. Caglar, S. H. Caglar, E. H.Twizell, The numerical solution of third-order boundary-value problems with fourth-degree B-spline functions. Int. J. Comput. Math.71 (1999) 373-381.

[6] H. N. Caglar, S.H. Caglar, E.H, Twizell, The numerical solution of fifth-order boundary-value problems with sixth-degree B-spline functions, Appl. Math. Lett. 12 (1999)25-30.

[7] H. N. Caglar, H. Caglar, B. Cagal, Spline solution of nonlinear beam problems, J. Concr. Applic. Math. 1 (3) (2003) 253-259.

[8] H. Caglar, N. Chaglar, K. Elfaituri. "B-spline interpolation compared with finite difference, finite element and finite volume methods which applied to two-point boundary value problem". J. Compute. Appl. Math. 175 (2006) 72-79.

[9] S. C. Chapra, R. P. Canale. "Numerical Method for Engineers", 3-th edition, WCB/McGraw-Hill, (1998), TA345.C47. [8] C. De Boor. (1978). “A Practical Guide to Splines” Springer-Verlag. New York.

[10] B. P. Demidovich, I. A. Maron (1987), "COMPUTATIONAL MATHEMATICS" (translated from Russian by G.Yankovsky) Mir Publishers, Moscow.

[11] E. P. Doolan, J. J. H. Miller, W. H. A. Schild, "Uniform Numerical methods for Problems with Initial and Boundary Layer", Boole Press, Ireland 1980.

[12] J. EIAN and J. Colin, "NUMERICAL ANALYSIS", (1992), (Arabic translation by Dr. Ahmed A.), Alfatih University, Tripoli-Libya.

[13] K. Elfaituri, L. Lojos, "Interpolation Theory", MSc. Thesis, Otvos Lurant University, (1998).

[14] Q. Fang, T. Tsuchiya, and T. Yamamoto, “Finite Difference, finite element, and finite volume methods applied to two-point boundary value problem”, J. Compute. Appl. Math. 139 (2002) 9-19.

[15] H. B. KELLER (1968), "Numerical method for two-point boundary-value problem". Blaisdell, Waltham, Mass., 184 pp. QA372.K42.

[16] E. Part-Enander, A. Sjoberg, B. Melin, P. Isaksson, "The Matlab Handbook", Addison Wesley Longman, (1996).

[17] L. B. Richard and J. Douglas Faires (1988), "Numerical Analysis". Prindle, Weber & Schmidt, Boston. QA297. B84.

[18] htt//: www.wikipedia.com

Appendix of Programs

```
function[u,v]=kutta4(x,u,v,h)
k1=fkota(x,u,v);                  l1=gkota(x,u,v);
k2=fkota(x+h/2,u+h*k1/2,v+h*l1/2); l2=gkota(x+h/2,u+h*k1/2,v+h*l1/2);
k3=fkota(x+h/2,u+h*k2/2,v+h*l2/2); l3=gkota(x+h/2,u+h*k2/2,v+h*l2/2);
k4=fkota(x+h,u+h*k3,v+h*l3);      l4=gkota(x+h,u+h*k3,v+h*l3);
u=u+(h*[k1+2*k2+2*k3+k4]/6);      v=v+(h*[l1+2*l2+2*l3+l4]/6);
```

Program (kutta4.m) applies for four steps Runge-Kutta method.

```
function [d1,err]=spl2(d,f,n,a,b,x)
if nargin==5;h=(b-a)/n;x=a:h:b; else x=[a x b];end;y=feval(f,x);
h=(b-a)/50;xx=a:h:b;yy=feval(f,xx);ma=max(yy);mi=min(yy);
l=(ma-mi)/2; axis([a,b,mi-l,ma+l]); plot(xx,yy,'g');hold on;
pause; plot(x,y,'*'); pause;d1=d; xx=a; yy=feval(f,a);err=0;
for i=1:n
    x1=x(i);x2=x(i+1);h=x2-x1;
    f1=feval(f,x1);f2=feval(f,x2);
    p=[(f2-f1-d1*h)/h.^2,d1,f1]; hh=h/20; t=x1+hh*[1:20];
    z=polyval(p,t-x1);
    w=feval(f,t); hi=max(abs(z-w)); err=max(hi,err);xx=[xx t];
    yy=[yy z];
    d1=2*p(1)*h+p(2);end,
plot(xx,yy,'b');pause,clg;hold off
```

Program (Spl2.m) which is a matlab program interpolate quadratic spline polynomial that passes'' n'' nodes in [a, b] by known of $f'(x_0)$ = d.

Curriculum Vitae (CV)

I finished B.Sc. in 1984 from Alfatih University Tripoli-Libya.

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